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Electrical conductivity functions in the magnetotelluric and magnetovariation methods

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Dedicated to the memory of Prof. Louis Cagniard

RESUME. – *Les auteurs développent une théorie établissant une relation fonctionnelle linéaire entre les composantes du champ électromagnétique terrestre. Ces relations existent si les champs observés sont bien représentés par des modèles de classes spéciales appelées classes algébriques. Ces dernières sont créées par les champs de dipôles magnétiques et électriques, des ondes planes, des champs magnétique et électrique uniformes. Les coefficients des relations linéaires dans une classe algébrique ne dépendent pas de la polarisation et de l'intensité des sources du champ et peuvent être considérés comme des fonctions de la conductivité électrique qui reflètent la distribution de conductivité dans la Terre. Ces coefficients forment les matrices magnéto-telluriques (impédance, admittance, tellurique et magnétique) utilisées en sondages ou profils de magnétotellurique ou de magnétovariation et dans la méthode des courants telluriques. La forme de ces matrices dépend de la dimension de l'espace des vectors caractéristiques qui déterminent les courants extrinsèques ou les champs primaires.*

ABSTRACT. – *The authors develop a theory establishing a functional linear relationship between the components of the Earth's electromagnetic field. These relationships exist if the observed fields can be well approximated by model fields of special classes called the algebraic classes. The algebraic classes are created by the fields of electrical and magnetic dipoles, plane waves, uniform magnetic and electrical fields. The coefficients of the linear relations within an algebraic class do not depend on the polarisation and intensity of the field sources and can be looked upon as the electrical conductivity functions which reflect the conductivity distribution in the Earth. These coefficients compose the magnetotelluric matrices (impedance, admittance, telluric, magnetic) and induction matrices used in the magnetotelluric and magnetovariation sounding or profiling, and in telluric current methods. The shape of the magnetotelluric and induction matrices depends on the dimension of the space of the characteristic vectors which determine the extrinsic currents or the primary fields.*

Part I
MAGNETOTELLURIC MATRICES

Introduction

Magnetotelluric and magnetovariation methods are generally used for studying the Earth's electrical conductivity. They are based on the determination of the transfer functions which connect different components of a variable electromagnetic field excited by ionospheric and magnetospheric currents. These functions will be called the *electrical conductivity functions* as they indicate the conductivity distribution in the Earth. The main drawback is that the observations at one or a few stations form a rather too narrow space window allowing the determination of the type of excitation but not its detailed structure. Therefore for specifying the electrical conductivity functions we have to take resort to *stochastic method* which does not need any detailed information about the configuration of the external field. This method is directed toward the linear correlation between the field components and lies in the calculation of the electrical conductivity functions as the multiple linear regression coefficients

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(Schmucker, 1970). We know well that practical experience justifies such an approach, nevertheless, its one weak point is the lack of an adequate theory which could expose the *functional* nature of the linear relationships between the field components. Only some scattered fragments of this theory have been touched upon in the geophysical literature (Cantwell, 1960 ; Berdichewski, 1958, 1960, 1961, 1964, 1968 ; Wiese, 1965 ; Schmucker, 1964, 1970 ; Untiedt, 1964 ; Adam, 1964 ; Jankowski, 1972 ; Lilley and Bennet, 1973). Imperfection of the theory gives rise to many puzzles and disputes. In this paper an attempt has been made to develop a self-consistent theory establishing the functional linear relationship between the components of the Earth's electromagnetic field in a wide range of its variations (from the pulsations to world magnetic storms). The paper is in two separate parts : the first is devoted to the magnetotelluric matrices and the second to the induction matrices and the main geophysical applications.

Algebraic classes of electromagnetic fields

Maxwell s equations though linear. nonetheless, give a *differential* but not an *algebraic* relationship between the components of the electromagnetic field. Therefore, in considering some field, say, the polar electrojet field, we have no right to assume that the field components are interrelated by a functional linear relationship with coefficients not dependent on the position of the electrojet. *Postulation of such relationships needs to impose certain restrictions on the structure of the field sources.* These restrictions lead to special classes of electromagnetic fields which constitute the subject matter of our paper.

We shall confine ourselves to the consideration of the frequency domain supposing that Fourier analysis of the transient electromagnetic field (pulsations, bays, solar diurnal variations, world storms) has been carried out. At the frequency ω an electromagnetic field $\vec{E}, \vec{H}$ satisfies the equations :

$$\text{rot}\, \vec{H} = \sigma^* \vec{E} + \vec{j}_{ex}$$

$$\text{rot}\, \vec{E} = i\omega\mu\vec{H}$$

where σ^* is the complex electrical conductivity, μ is the magnetic permeability and $\vec{j}_{ex}$ is the density of extrinsic electrical current.

The starting-point of our theory is *the field source*, i.e. the extrinsic current. The function $\vec{j}_{ex}$ can always be represented by means of a linear operator $\mathfrak{A}$ acting on some free vector $\vec{M}$ (real or complex) :

$$\vec{j}_{ex}(\vec{r}) = \mathfrak{A}(\vec{r})\vec{M} \qquad (1a)$$

This vector $\vec{M}$ does not depend on the space coordinates. It characterizes the *polarisation* and the *intensity* of the extrinsic current and will be called the *characteristic vector of the field.* The operator $\mathfrak{A}$ does depend on the space coordinates and determines the *geometry* of extrinsic current, i.e. the type of excitation. We shall call it the *excitation operator* or simply the *excitation.*

In general the choice of representation (1a) is arbitrary. Considering some model field, we shall choose the representation providing the maximum simplicity and the physical lucidity.

Let's take some sequence of the extrinsic currents with the same geometry but different polarisation and intensity. To these extrinsic currents correspond distributions $\vec{j}_{ex}$, obtained by the action of the same excitation operator on different characteristic vectors. In the following pages we shall show that in a given medium the components of the electromagnetic fields excited by such extrinsic currents are interconnected by linear algebraic relationships, whose coefficients are the same for all distributions $\vec{j}_{ex}$ under consideration. These coefficients are connected with the type of excitation and depend on the frequency, the observation site and the properties of the medium. The set of such electromagnetic fields will be called the *algebraic class.*

Thus, *the electromagnetic fields whose characteristic vectors are transformed into extrinsic currents by the same excitation operator form the algebraic class.* To each excitation operator $\mathfrak{A}$ corresponds its own algebraic class with the associated set of characteristic vectors $\vec{M}$ which compose some linear space $\mathfrak{M}_N$ of dimension N. The value N can vary from 1 to 3 depending on the polarisation of the extrinsic current. In every space $\mathfrak{M}_N$ there are N linearly independent vectors $\vec{M}$ corresponding to extrinsic currents with different polarisation.

In geoelectrics we often deal with the models where the primary fields $\vec{E}^P, \vec{H}^P$ but not the extrinsic currents are given. Any primary field can likewise be expressed as a linear transform of the characteristic vector $\vec{M}$:

$$\vec{E}^P(\vec{r}) = b^E(\vec{r})\vec{M} \quad \vec{H}^P(\vec{r}) = b^H(\vec{r})\vec{M}, \qquad (1b)$$

where b^E and b^H are the excitation operators. The set of electromagnetic fields whose characteristic vectors are transformed into primary fields by the same excitation operators forms an algebraic class.

Let us give some examples of fields which form the algebraic classes.

1/ All electromagnetic fields excited by extrinsic currents of identical geometry and polarisation form an algebraic class with one-dimensional characteristic vector space. For example, let's consider the field of an infinitely long rectilinear current J flowing along the axis x_1 of a Cartesian system $x_1x_2x_3$. The x_1-direction is defined by the unit vector $\vec{d}_1$. In this model

$$\vec{j}_{ex}(\vec{r}) = J\delta(x_2)\,\delta(x_3)\,\vec{d}_1$$

where δ is the Dirac function. According to (1a) we set

$$\vec{M} = J\vec{d}_1 \qquad \mathcal{C}(\vec{r}) = \delta(x_2)\,\delta(x_3)$$

The characteristic vectors $\vec{M}$ determine the current strength and are always parallel with the current line. The excitation $\mathcal{C}$ depends on the position of the current line. All fields excited in a heterogeneous medium by an infinitely long rectilinear current flowing along a given line belong to the same algebraic class with the space $\mathfrak{M}_1$.

2/ Consider now an example of algebraic class with the space $\mathfrak{M}_2$. Let the model consist of outer homogeneous and inner heterogeneous domains. Take the primary field in the form of an arbitrarily polarized uniform or non-uniform plane wave with amplitudes $\vec{E}_0$, $\vec{H}_0$ and wave vector $\vec{k}$:

$$\vec{E}^P(\vec{r}) = \vec{E}_0\, e^{\vec{k}.\vec{r}} \qquad \vec{H}^P(\vec{r}) = \vec{H}_0\, e^{-\vec{k}.\vec{r}}$$

The equation rot $\vec{H}^P = \sigma^* \vec{E}^P$ gives $\vec{E}_0 = Z\vec{H}_0$, where σ^* is the conductivity of the outer domain and Z is the impedance with the matrix $Z_{ij}(i,j = 1,2,3)$. In an arbitrary Cartesian basis $Z_{11} = Z_{22} = Z_{33} = 0$, $Z_{12} = -Z_{21} = k_3/\sigma^*$, $Z_{13} = -Z_{31} = -k_2/\sigma^*$.

Thus, according to (1b) we can write

$$\vec{M} = \vec{H}_0 \qquad b^E(\vec{r}) = Z e^{-\vec{k}.\vec{r}} \qquad b^H(\vec{r}) = e^{-\vec{k}.\vec{r}}$$

The characteristic vectors $\vec{M}$ determine the polarization and intensity of the waves. The excitation $b^{E,H}$ depends on the wave vector $\vec{k}$. For a given $\vec{k}$, the vectors $\vec{M}$ are the elements of the two-dimensional space $\mathfrak{M}_2$. Indeed, the equation div $\vec{H}^P = 0$ shows that $\vec{M}.\vec{\tilde{k}} = 0$, i.e. the vectors $\vec{M}$ are orthogonal to the conjugate wave vector. Thus, all electromagnetic fields excited in a heterogeneous medium by arbitrarily polarized uniform or non-uniform plane waves with identical wave vector belong to the same algebraic class with the space $\mathfrak{M}_2$.

3/ The fields of electrical or magnetic dipoles located at a given point and primarily uniform magnetic or electrical fields are examples of algebraic class in which the characteristic vectors may compose a three-dimensional space.

Let us place an arbitrarily oriented electrical dipole of moment $\vec{P}$ at a given point $\vec{r}_0$ in a heterogeneous medium. In this model

$$\vec{j}_{ex}(\vec{r}) = \delta(\vec{r} - \vec{r}_0)\,\vec{P}$$

Hence, we can choose the vector $\vec{M} = \vec{P}$ as the characteristic vector with the excitation $\mathcal{C} = \delta(\vec{r} - \vec{r}_0)$. Evidently, the characteristic vectors are arbitrarily oriented and the excitation depends on the position of the dipole centre. Consequently, all electromagnetic fields excited in a heterogeneous medium by arbitrarily oriented electrical dipoles with a common centre belong to the same algebraic class with the space $\mathfrak{M}_3$.

Fields of magnetic dipoles form an analogous algebraic class. This can be easily demonstrated by substituting the magnetic currents for the extrinsic electrical currents.

Models with primarily uniform magnetic fields are used for investigating the electromagnetic induction in conducting bodies surrounded by a non-conducting medium. In these models

$$\vec{H}^P(r) = \vec{H}_0,$$

where $\vec{H}_0$ is a constant vector. We shall take that

$$\vec{M} = \vec{H}_0 \qquad b^H(\vec{r}) = 1$$

The characteristic vectors are arbitrarily oriented and the excitation is constant. Consequently, all electromagnetic fields excited in a heterogeneous conducting body by a uniform magnetic field belong to the same algebraic class with the space $\mathfrak{M}_3$.

A similar algebraic class is generated by the primarily uniform electrical fields with $\omega = 0$.

The algebraic classes with the space $\mathfrak{M}_3$ may also be generated by more complex fields. For instance, the following example deserves our attention. Let an arbitrarily oriented electrical dipole of moment $\vec{P}(t) = f(t)\,\vec{p}$ move with constant velocity V along the axis x_1 where $f(t)$ is some time function, and $\vec{p}$ is a constant unit vector. In this model

$$\vec{j}_{ex}(\vec{r}, t) = f(t)\,\delta(x_1 - Vt)\,\delta(x_2)\,\delta(x_3)\,\vec{p}$$

After applying the Fourier transform we obtain the extrinsic current in frequency domain:

$$\vec{j}_{ex}(\vec{r}) = \int_{-\infty}^{+\infty} \vec{j}_{ex}(\vec{r}, t)\, e^{i\omega t}\, dt = \frac{1}{V} f\!\left(\frac{x_1}{V}\right) e^{i\frac{\omega x_1}{V}}\, \delta(x_2)\,\delta(x_3)\,\vec{p}$$

For the characteristic vector we shall take $\vec{M} = \vec{P}(t_0)$ where t_0 is the time so chosen that $\vec{P}(t_0) \neq 0$. In this case:

$$\mathcal{C}(\vec{r}) = \frac{1}{Vf(t_0)}\, f\!\left(\frac{x_1}{V}\right) e^{i\frac{\omega x_1}{V}}\, \delta(x_2)\,\delta(x_3)$$

Thus, the characteristic vectors determine the orientation and the intensity of the dipole. The excitation depends on the position of the trajectory, velocity of the dipole, and the law of relative variation of its moment. Consequently, the electromagnetic fields excited in a heterogenerous medium by arbitrarily oriented electrical dipoles moving along a given straight trajectory with a given velocity and a given law for the relative variation of its moment generate an algebraic class with the space $\mathfrak{M}_3$. An analogous class is obtained in case of magnetic dipoles.

It is obvious that we can reduce the dimension of the characteristic vector space for each algebraic class. For example, the horizontal dipoles generate a class with the space $\mathfrak{M}_2$, whereas the linearly polarized plane waves create a class with the space $\mathfrak{M}_1$.

We shall confine ourselves to these examples of algebraic classes which are very suitable for the simulation of the Earth's electromagnetic field.

Electromagnetic field as a linear transform of characteristic vectors

Electromagnetic field is a linear transform of the extrinsic current :

$$\vec{E}(\vec{r}) = G^E(\vec{r} - \vec{r}_1) * \vec{j}_{ex}(\vec{r}_1)$$
$$\vec{H}(\vec{r}) = G^H(\vec{r} - \vec{r}_1) * \vec{j}_{ex}(\vec{r}_1),$$

where the asterisk denotes convolution with respect to the variable $\vec{r}_1$, and G^E, G^H are derived Green's functions. They obviously depend on the frequency ω and the distribution of electromagnetic properties of the medium.

Let the field belong to some algebraic class with the excitation $\mathfrak{A}$. Then by virtue of (1a) we have

$$\vec{E}(\vec{r}) = e(\vec{r})\vec{M} \quad (a) \qquad \vec{H}(\vec{r}) = h(\vec{r})\vec{M} \quad (b) \quad (2)$$

where

$$e(\vec{r}) = G^E(\vec{r} - \vec{r}_1) * \mathfrak{A}(\vec{r}_1)$$
$$h(\vec{r}) = G^H(\vec{r} - \vec{r}_1) * \mathfrak{A}(\vec{r}_1)$$

The relationship (2) virtually means that the electrodynamic problem can be broken down into several independent problems whose number is exactly equal to the dimension of the space $\mathfrak{M}_N$.

We shall call the linear operators e and h the *characteristic operators of the field*. Analogous operators associated with the excitation operators b^E and b^H are obtained in the models with given primary fields (the action of primary field through linear operations can be reduced to the action of real or fictitious extrinsic currents).

To each algebraic class correspond its own operators e and h which depend on the *frequency*, the *observation site* and the *distribution of the electromagnetic properties of the medium*. By means of the operators e and h we can transform at every point $\vec{r}$ the vector $\vec{M}$ which is an element of the space $\mathfrak{M}_N$ into vectors $\vec{E}$ and $\vec{H}$ which are the elements of the spaces $\mathcal{E}_K$ and $\mathcal{H}_L$, respectively, where K, L are the dimension of these spaces. To each point $\vec{r}$ there correspond its own spaces $\mathcal{E}_K$ and $\mathcal{H}_L$.

Depending on the properties of the medium the spaces $\mathcal{E}_K$ and $\mathcal{H}_L$ may retain the dimension of the space $\mathfrak{M}_N$ (*mutually unique transformation*) or have lesser dimension (*degenerate transformation*).

The mutually unique transformation is carried out by *reversible* operators. In this case N characteristic vectors $\vec{M}$ from the space $\mathfrak{M}_N$ corresponding to the extrinsic currents or primary fields with different polarisation are linearly independent and generate N linearly independent vectors $\vec{E}$ from the space $\mathcal{E}_K$ ($K = N$) and N linearly independent vectors $\vec{H}$ from the space $\mathcal{H}_L$ ($L = N$).

The degenerate transformation is carried out by *irreversible* operators. In this case the number of linearly independent vectors $\vec{E}$ or $\vec{H}$ is less than the number of linearly independent vectors $\vec{M}$.

We shall give two examples of degenerate transformation.

Let a quasi-stationary electromagnetic field from the algebraic class with the space $\mathfrak{M}_3$ excite a body of finite conductivity surrounded by an insulator. At the inner surface of the body we have $\vec{E}.\vec{n} = 0$ where $\vec{n}$ is the unit normal vector. At the points of this surface the field $\vec{E}$ is always polarized in the tangent plane and generates the space $\mathcal{E}_2$. Therefore, the mapping of $\mathfrak{M}_3$ into $\mathcal{E}_2$ is a degenerate transform, and consequently the operator e is irreversible.

Now consider a model in which a perfect conductor is surrounded by a medium of finite conductivity. Let this model be excited by an electromagnetic field from the algebraic class with the space $\mathfrak{M}_3$. On the surface of the conductor we have $\vec{H}.\vec{n} = 0$ and $\vec{n} \times [\vec{n} \times \vec{E}] = 0$. At the points of this surface the field $\vec{H}$ is always polarized in the tangent plane and generates the space $\mathcal{H}_2$, while the field $\vec{E}$ is polarized normal to the conductor surface and generates the space $\mathcal{E}_1$. Therefore, the mapping of $\mathfrak{M}_3$ into $\mathcal{H}_2$ and $\mathcal{E}_1$ is degenerate and the operators e and h are irreversible.

The characteristic operators of the field can be expressed in the form of matrices, which we shall call as the *fundamental characteristic matrices*. By deleting one or two rows from these fundamental matrices we obtain the so-called *reduced matrices* which will be more convenient for investigating the fields with the characteristic vector spaces $\mathfrak{M}_2$ and $\mathfrak{M}_1$.

Magnetotelluric operators

We shall take an electromagnetic field from some algebraic class and examine three cases.

1/ Let the operators e and h be both reversible in some domain $\mathcal{V}$ (i.e. the inverse operators e^{-1} and h^{-1} exist). Then from (2) we get

$$\vec{M} = \begin{cases} e^{-1}(\vec{r})\,\vec{E}(\vec{r}) & \text{(a)} \\ h^{-1}(\vec{r})\,\vec{H}(\vec{r}) & \text{(b)} \end{cases} \qquad (3)$$

Substituting (3b) into (2a) and (3a) into (2b) we obtain

$$\vec{E}(\vec{r}) = Z(\vec{r})\,\vec{H}(\vec{r}) \quad \text{(a)} \quad \vec{H}(\vec{r}) = Y(\vec{r})\,\vec{E}(\vec{r}) \text{ (b)} \quad (4)$$

where Z and Y are operators which represent the *impedance* and *admittance* respectively :

$$Z(\vec{r}) = e(\vec{r})h^{-1}(\vec{r}) \quad Y(\vec{r}) = h(\vec{r})e^{-1}(\vec{r}) \quad (5)$$

Considering $\vec{E}$ and $\vec{H}$ at different points $\vec{r}_1$ and $\vec{r}_2$ belonging to the domain $\mathcal{V}$, we get

$$\begin{aligned} \vec{E}(\vec{r}_1) &= t(\vec{r}_1,\vec{r}_2)\,\vec{E}(\vec{r}_2) \quad \text{(a)} \\ H(r_1) &= m(\vec{r}_1,\vec{r}_2)\,\vec{H}(\vec{r}_2) \quad \text{(b)} \end{aligned} \qquad (6)$$

where t and m are the *telluric* and *magnetic* operators :

$$t(\vec{r}_1,\vec{r}_2) = e(\vec{r}_1)\,e^{-1}(\vec{r}_2)$$

$$m(\vec{r}_1,\vec{r}_2) = h(\vec{r}_1)\,h^{-1}(\vec{r}_2) \quad (7)$$

The operators Z, Y, m, t will be called the *magnetotelluric operators*. These four operators transform the electrical field into the magnetic field and vice versa. *To each algebraic class there correspond its own operators Z, Y, t, m which depend on the frequency, the observation site and the medium.*

Evidently, all magnetotelluric operators are reversible in the domain $\mathcal{V}$:

$$Z(\vec{r}) = Y^{-1}(\vec{r}) \qquad Y(\vec{r}) = Z^{-1}(\vec{r})$$

$$t(\vec{r}_2,\vec{r}_1) = t^{-1}(\vec{r}_1,\vec{r}_2) \quad m(\vec{r}_2,\vec{r}_1) = m^{-1}(\vec{r}_1,\vec{r}_2) \quad (8)$$

The following relations hold between the magnetotelluric operators :

$$\begin{aligned} Z(\vec{r}_1) &= t(\vec{r}_1,\vec{r}_2)\,Z(\vec{r}_2)\,m(\vec{r}_2,\vec{r}_1) \\ Y(\vec{r}_1) &= m(\vec{r}_1,\vec{r}_2)\,Y(\vec{r}_2)\,t(\vec{r}_2,\vec{r}_1) \\ t(\vec{r}_1,\vec{r}_2) &= Z(\vec{r}_1)\,m(\vec{r}_1,\vec{r}_2)\,Y(\vec{r}_2) \\ m(\vec{r}_1,\vec{r}_2) &= Y(\vec{r}_1)\,t(\vec{r}_1,\vec{r}_2)\,Z(\vec{r}_2) \end{aligned} \qquad (9)$$

2/ If one of the characteristic operators of the field is reversible and the other is irreversible in the domain $\mathcal{V}$, then only two magnetotelluric operators exist ; we have the operators Y and t when e is reversible, or otherwise the operators Z and m when h is reversible. Consequently the systems (4) and (6) of direct and inverse transformations are partially decayed. The transforms (4b) and (6a) hold true when e is reversible, whereas the transforms (4a) and (6b) hold true when h is reversible. Impedance and admittance operators become irreversible and the relationship (9) ceases to be valid.

3/ If both the characteristic operators of the field are irreversible in the domain $\mathcal{V}$, then the magnetotelluric operators do not exist and the systems of transformations (4) and (6) completely decay.

The conditions for the existence and the reversibility of the magnetotelluric operators are tabulated below.

Table 1

Conditions for the existence and the reversibility of magnetotelluric operators

e \ h	Reversible	Irreversible
Reversible	Reversible operators Z, Y, t, m exist	Irreversible operator Y and reversible operator t exist
Irreversible	Irreversible operators Z and reversible operator m exist	Magnetotelluric operators do not exist

The matrices of magnetotelluric operators will be called the *fundamental magnetotelluric matrices.* By deleting one or two rows from these matrices we obtain the *reduced magnetotelluric matrices.* The shape of the magnetotelluric matrices depends on the dimension of the characteristic vector space. *The elements of the magnetotelluric matrices are the electrical conductivity functions.*

Magnetotelluric matrices in the class with the space $\mathcal{M}_3$

The characteristic operators e and h act from the space $\mathcal{M}_3$ into the spaces $\mathcal{E}_K$ and $\mathcal{H}_L$ $(K, L \leqslant 3)$. We shall embed the spaces $\mathcal{M}_3$, $\mathcal{E}_K$, $\mathcal{H}_L$ into a three-dimensional physical space $\mathcal{D}_3$. Now the operators e and h act in the space $\mathcal{D}_3$. In an arbitrary Cartesian basis $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ of this space they are described by the fundamental matrices e_{ij} and h_{ij} $(i, j = 1, 2, 3)$, i.e. by the square matrices of order 3.

The operators e and h are reversible if the determinants of their matrices are non-zero. This is the necessary condition for the existence of the magnetotelluric matrices (Table I).

Let e and h be reversible operators. Using the elements e_{ij} and h_{ij} it is a simple matter to get the elements of the magnetotelluric matrices. By virtue of (5) and (7) we can write

$$\begin{aligned} Z_{ij}(\vec{r}) &= e_{ik}(\vec{r})\, h^{-1}_{kj}(\vec{r}) \\ Y_{ij}(\vec{r}) &= h_{ik}(\vec{r})\, e^{-1}_{kj}(\vec{r}) \\ t_{ij}(\vec{r}_1, \vec{r}_2) &= e_{ik}(\vec{r}_1)\, e^{-1}_{kj}(\vec{r}_2) \\ m_{ij}(\vec{r}_1, \vec{r}_2) &= h_{ik}(\vec{r}_1)\, h^{-1}_{kj}(\vec{r}_2) \end{aligned} \tag{10}$$

$$i, j, k = 1, 2, 3.$$

Thus, we have four fundamental magnetotelluric square matrices of order 3 (impedance, admittance, telluric and magnetic matrices) :

$$Z_{ij}, Y_{ij}, t_{ij}, m_{ij}\ (i, j = 1, 2, 3).$$

These matrices are reversible as their determinants are equal to the product of the non-zero determinants of the corresponding direct and inverse characteristic matrices. The elements of the magnetotelluric matrices depend on the frequency, the observation site and the distribution of the electromagnetic properties of the medium. Within a given class they can be treated as the electrical conductivity functions which indicate the geoelectrical structure of the medium.

Impedance, admittance, telluric and *magnetic tensors* correspond to the fundamental magnetotelluric matrices. These tensors act in the space $\mathcal{D}_3$. From (4) and (6) we have

$$\begin{aligned} E_i(\vec{r}) &= Z_{ij}(\vec{r})\, H_j(\vec{r}) \\ H_i(\vec{r}) &= Y_{ij}(\vec{r})\, E_j(\vec{r}) \\ E_i(\vec{r}_1) &= t_{ij}(\vec{r}_1, \vec{r}_2)\, E_j(\vec{r}_2) \\ H_i(\vec{r}_1) &= m_{ij}(\vec{r}_1, \vec{r}_2)\, H_j(\vec{r}_2) \end{aligned} \tag{11}$$

$$i, j = 1, 2, 3$$

where E_i and H_i are the Cartesian components of the vectors $\vec{E}$ and $\vec{H}$. The tensors are interconnected according to (8) and (9).

If only one of the characteristic operators is reversible then there exist only two magnetotelluric matrices, namely, the impedance and the magnetic matrices when h is reversible or the admittance and the telluric matrices when e is reversible. For example, let the operator h be reversible, then we have the irreversible (degenerate) impedance matrix with a rank equal to the dimension of the space $\mathcal{E}_K$, and the reversible magnetic matrix.

Finally, if both the characteristic operators are irreversible, then the magnetotelluric matrices do not exist. Fortunately, such an unfavourable situation does not occur in the practical geophysical investigations.

Thus, we can find the electrical conductivity functions if the magnetotelluric matrices exist. These matrices, irrespective of their ranks, can be determined uniquely. As an example consider the impedance matrix. At the point $\vec{r}$ take three linearly independent vectors $\vec{H}^{(1)}, \vec{H}^{(2)}, \vec{H}^{(3)}$ with the corresponding vectors $\vec{E}^{(1)}, \vec{E}^{(2)}, \vec{E}^{(3)}$ (linearly independent or dependent). Then by (11) we get

$$E_i^{(k)}(\vec{r}) = Z_{ij}(\vec{r})\, H_j^{(k)}(\vec{r})$$

$$i, j, k = 1, 2, 3$$

Obviously the matrix $H_j^{(k)}$ formed by the components of three linearly independent vectors is non-degenerate, i.e. its inverse $[H_j^{(k)}]^{-1}$ exists. Therefore,

$$Z_{ij}(\vec{r}) = E_i^{(k)}(\vec{r})\, [H_j^{(k)}(\vec{r})]^{-1} \quad i, j, k = 1,2,3 \tag{12}$$

Other magnetotelluric matrices (matrices of electrical conductivity functions) are determined in a similar manner.

Magnetotelluric matrices in the class with the space $\mathcal{M}_2$

When the spaces $\mathcal{M}_2$ and $\mathcal{E}_K$, $\mathcal{H}_L$ $(K, L \leqslant 2)$ are embedded into the three-dimensional physical space $\mathcal{D}_3$ we formally obtain square matrices of order 3. These matrices cannot be determined uniquely and may prove to be degenerate even if the operators are reversible. We shall overcome this difficulty by expressing the relationship between the vectors $\vec{E}$ and $\vec{H}$ in terms of the fundamental rectangular matrice 3 x 2 and reduced square matrices of order 2 acting on linearly independent vector components.

Let $\{\vec{m}_1, \vec{m}_2\}$ be the basis of the space $\mathcal{M}_2$ and expand the vector $\vec{M}$ in terms of this basis :

$$\vec{M} = M_j \vec{m}_j \quad j = 1, 2$$

Then embed the spaces $\mathcal{E}_K$ and $\mathcal{H}_L$ $(K, L \leqslant 2)$ into a three-dimensional physical space $\mathcal{D}_3$ with arbitrary Cartesian basis $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$. Expanding the vectors $\vec{E}$ and $\vec{H}$ in terms of this basis, we can write :

$$\vec{E} = E_i \vec{d}_i \quad \vec{H} = H_i \vec{d}_i \quad i = 1, 2, 3.$$

Now the characteristic operators e and h act from the space $\mathcal{M}_2$ into the space $\mathcal{D}_3$:

$$\begin{aligned} \vec{E}(\vec{r}) &= e(\vec{r})\vec{M} = M_j\, e(\vec{r})\, \vec{m}_j = M_j\, \vec{e}_j(\vec{r}) \quad \text{(a)} \\ \vec{H}(\vec{r}) &= h(\vec{r})\vec{M} = M_j\, h(\vec{r})\, \vec{m}_j = M_j\, \vec{h}_j(\vec{r}) \quad \text{(b)} \end{aligned} \tag{13}$$

$$j = 1, 2$$

where

$$\vec{e}_j(\vec{r}) = e(\vec{r})\vec{m}_j \quad \vec{h}_j(\vec{r}) = h(\vec{r})\vec{m}_j\,.$$

The vectors $\vec{e}_j$ and $\vec{h}_j$ are the elements of the space $\mathcal{D}_3$ and thus can be expressed in terms of the basis $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$:

$$\vec{e}_j(\vec{r}) = e_{ij}(\vec{r})\,\vec{d}_i \quad \vec{h}_j(\vec{r}) = h_{ij}(\vec{r})\,\vec{d}_i$$

$$i = 1, 2, 3 \qquad j = 1, 2$$

Thus, we have

$$E_i(\vec{r}) = e_{ij}(\vec{r})\,M_j \quad H_i(\vec{r}) = h_{ij}(\vec{r})M_j$$

$$i = 1, 2, 3 \qquad j = 1, 2$$

where e_{ij} and h_{ij} are the fundamental characteristic matrices 3×2 of the operators e and h in the bases

$$\{\vec{d}_1, \vec{d}_2, \vec{d}_3\} \quad \text{and} \quad \{\vec{m}_1, \vec{m}_2\}.$$

If the operators e and h are reversible, the rank of their rectangular matrices is 2.

Now return to the Table 1 which shows the conditions for the existence and the reversibility of the magnetotelluric operators.

Let the operators e and h be reversible, i.e. the rank of their matrices is 2. Then from each rectangular matrix by deleting some one row we can form at least one reduced square matrix of order 2 with a non-zero determinant. Assume that such matrices are $e_{\hat{i}j}$ and $h_{\hat{i}j}$ ($\hat{i}, \hat{j} = 1, 2$). These matrices are formed by the first and the second rows of the rectangular matrices e_{ij} and h_{ij}. Their determinants are the basic minors q_3^e and q_3^h of the matrices e_{ij} and h_{ij} (the subscript in the minor indicates the number of the deleted row).

The reduced matrices $e_{\hat{i}\hat{j}}$ and $h_{\hat{i}\hat{j}}$ correspond to the operators $\tilde{e}$ and $\tilde{h}$ which transform the vector $\vec{M}$ into the vectors $\vec{E}_{12}$ and $\vec{H}_{12}$:

$$\vec{E}_{12}(\vec{r}) = \tilde{e}(\vec{r})\vec{M} \quad \vec{H}_{12}(\vec{r}) = \tilde{h}(\vec{r})\vec{M}$$

Obviously, the vectors $\vec{E}_{12}$ and $\vec{H}_{12}$ have linearly independent components and are the projections of the vectors $\vec{E}$ and $\vec{H}$ on the plane $x_1\ x_2$:

$$\vec{E}_{12}(\vec{r}) = E_{\hat{i}}(\vec{r})\vec{d}_{\hat{i}} \quad \vec{H}_{12}(\vec{r}) = H_{\hat{i}}(\vec{r})\,\vec{d}_{\hat{i}} \quad \hat{i} = 1, 2$$

The operators $\tilde{e}$ and $\tilde{h}$ are reversible because $q_3^e \neq 0$ and $q_3^h \neq 0$. Therefore,

$$\vec{M} = \begin{cases} \tilde{e}^{-1}(\vec{r})\vec{E}_{12}(\vec{r}) & \text{(a)} \\ \tilde{h}^{-1}(\vec{r})\vec{H}_{12}(\vec{r}) & \text{(b)} \end{cases} \tag{14}$$

Substituting (14b) into (13a) and (14a) into (13b) we get

$$\vec{E}(\vec{r}) = Z(\vec{r})\vec{H}_{12}(\vec{r}) \quad \vec{H}(\vec{r}) = Y(\vec{r})\vec{E}_{12}(\vec{r})$$

where

$$Z(\vec{r}) = e(\vec{r})\,\tilde{h}^{-1}(\vec{r}) \quad Y(\vec{r}) = h(\vec{r})\tilde{e}^{-1}(\vec{r})$$

Similarly,

$$\vec{E}(\vec{r}_1) = t(\vec{r}_1, \vec{r}_2)\,\vec{E}_{12}(\vec{r}_2)$$
$$\vec{H}(\vec{r}_1) = m(\vec{r}_1, \vec{r}_2)\,\vec{H}_{12}(\vec{r}_2)$$

where

$$t(\vec{r}_1, \vec{r}_2) = e(\vec{r}_1)\,\tilde{e}^{-1}(\vec{r}_2)$$
$$m(\vec{r}_1, \vec{r}_2) = h(\vec{r}_1)\,\tilde{h}^{-1}(\vec{r}_2)$$

The operators Z, Y, t and m act from the plane $x_1\,x_2$ into the three-dimensional space, and transform the projections of the electrical or magnetic fields into the total electrical or magnetic field.

In the Cartesian basis $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ we have

$$E_i(\vec{r}) = Z_{ij}(\vec{r})\,H_j(\vec{r}) \quad H_i(\vec{r}) = Y_{ij}(\vec{r})\,E_j(\vec{r})$$
$$E_i(\vec{r}_1) = t_{ij}(\vec{r}_1, \vec{r}_2)\,E_j(\vec{r}_2) \tag{15}$$
$$H_i(\vec{r}_1) = m_{ij}(\vec{r}_1, \vec{r}_2)H_j(\vec{r}_2)$$
$$i = 1, 2, 3 \qquad j = 1, 2$$

where

$$Z_{i\hat{j}}(\vec{r}) = e_{i\hat{k}}(\vec{r})\,\tilde{h}^{-1}_{\hat{k}\hat{j}}(\vec{r}) \quad Y_{i\hat{j}}(\vec{r}) = h_{i\hat{k}}(\vec{r})\,\tilde{e}^{-1}_{\hat{k}\hat{j}}(\vec{r})$$
$$t_{i\hat{j}}(\vec{r}_1, \vec{r}_2) = e_{i\hat{k}}(\vec{r}_1)\,\tilde{e}^{-1}_{\hat{k}\hat{j}}(\vec{r}_2) \tag{16}$$
$$m_{i\hat{j}}(\vec{r}_1, \vec{r}_2) = h_{i\hat{k}}(\vec{r}_1)\,\tilde{h}^{-1}_{\hat{k}\hat{j}}(\vec{r}_2)$$
$$i = 1, 2, 3 \qquad \hat{j}, \hat{k} = 1, 2$$

The matrices $Z_{ij}, Y_{ij}, t_{ij}, m_{ij}$ ($i = 1, 2, 3$; $j = 1, 2$) are the fundamental magnetotelluric matrices 3×2 in the class with the space $\mathfrak{M}_2$.

On deleting the third row from each matrix we obtain the reduced matrices $Z_{\hat{i}\hat{j}}, Y_{\hat{i}\hat{j}}, t_{\hat{i}\hat{j}}, m_{\hat{i}\hat{j}}$ ($\hat{i}, \hat{j} = 1, 2$). The operators $\tilde{Z}, \tilde{Y}, \tilde{t}, \tilde{m}$ correspond to these square matrices of order 2. They are reversible, and consequently satisfy the relationships (8) and (9). They act in the plane $x_1 x_2$ and transform the E, H projections into the E, H projections. The impedance, admittance, telluric and magnetic tensors acting in the same plane $x_1 x_2$ correspond to these operators :

$$E_{\hat{i}}(\vec{r}) = Z_{\hat{i}\hat{j}}(\vec{r}) H_{\hat{j}}(\vec{r}) \qquad H_{\hat{i}}(\vec{r}) = Y_{\hat{i}\hat{j}}(\vec{r}) E_{\hat{j}}(\vec{r})$$

$$E_{\hat{i}}(\vec{r}_1) = t_{\hat{i}\hat{j}}(\vec{r}_1, \vec{r}_2) E_{\hat{j}}(\vec{r}_2) \tag{17}$$

$$H_{\hat{i}}(\vec{r}_1) = m_{\hat{i}\hat{j}}(\vec{r}_1, \vec{r}_2) H_{\hat{j}}(\vec{r}_2)$$

$$\hat{i}, \hat{j} = 1, 2$$

The magnetotelluric matrices (fundamental or reduced) can be determined uniquely. As an example, let us consider the impedance matrix. At the point $\vec{r}$ take two linearly independent vectors $\vec{H}^{(1)}$, $\vec{H}^{(2)}$ with the corresponding vectors $\vec{E}^{(1)}$, $\vec{E}^{(2)}$ (linearly independent or dependent). By virtue of (15) we have

$$E_i^{(\hat{k})}(\vec{r}) = Z_{i\hat{j}}(\vec{r}) H_{\hat{j}}^{(\hat{k})}(\vec{r})$$

$$i = 1, 2, 3 \qquad \hat{j}, \hat{k} = 1,2$$

The matrix $H_{\hat{j}}^{(\hat{k})}$ formed by the components of two linearly independent vectors is non-degenerate, i.e. its inverse $[H_{\hat{j}}^{(\hat{k})}]^{-1}$ exists. Therefore

$$Z_{i\hat{j}}(\vec{r}) = E_i^{(\hat{k})}(\vec{r}) [H_{\hat{j}}^{(\hat{k})}(\vec{r})]^{-1}$$

$$i = 1, 2, 3 \quad \hat{j}, \hat{k} = 1.2 \tag{18}$$

Thus, four fundamental magnetotelluric (impedance, admittance, telluric, and magnetic) matrices 3 x 2 exists for the reversible characteristic operators *e* and *h*. Each of these matrices can be reduced to a tensor matrix of order 2.

If only one of the characteristic operators is reversible, then there exist only two matrices, namely, the impedance and the magnetic matrices when the operator *h* is reversible or the admittance and the telluric matrices when the operator *e* is reversible.

No magnetotelluric matrices exist if both the characteristic operators are irreversible.

Magnetotelluric matrices in the class with the space $\mathfrak{M}_1$.

This case is so simple that it does not require any analysis. If $\vec{E} \neq 0$ and $\vec{H} \neq 0$, the characteristic operators *e* and *h* are always reversible. Their fundamental magnetotelluric matrices *Z*, *Y*, *t* and *m* are only column matrices.

On redusing the fundamental matrices we obtain the scalar proportionality coefficients :

$$E_i(\vec{r}) = Z_{ij}(\vec{r}) H_j(\vec{r}) \qquad H_i(\vec{r}) = Y_{ij}(\vec{r}) E_j(\vec{r})$$

$$E_i(\vec{r}_1) = t_{ij}(\vec{r}_1, \vec{r}_2) E_j(\vec{r}_2) \tag{19}$$

$$H_i(\vec{r}_1) = m_{ij}(\vec{r}_1, \vec{r}_2) H_j(\vec{r}_2)$$

$$i, j = 1, 2, 3$$

where the summation is not carried out with respect to the indexes *i* and *j*. Any two Cartesian components of the fields $\vec{E}$ and $\vec{H}$ are linearly dependent.

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PART II

INDUCTION MATRICES AND MAIN GEOPHYSICAL APPLICATIONS

Introduction

We shall recall the basic propositions of the Part I used in this Part. All the events are considered in the frequency domain. A *characteristic vector* is defined to be a free vector $\vec{M}$ which is transformed into extrinsic current or primary field by a linear operator (excitation operator). The set of all electromagnetic fields whose characteristic vectors are transformed into extrinsic current or primary field by one and the same excitation operator is called an *algebraic class*. Within a given algebraic class $\vec{H} = h\vec{M}$ where $\vec{H}$ is the magnetic field, and h is a linear operator called the *characteristic operator*. The characteristic operator h is connected with the type of excitation and depends on the frequency, the observation site and the distribution of electromagnetic properties of the medium. The operator h transforms the vector $\vec{M}$ being an element of the linear space $\mathfrak{M}_N$ into the vector $\vec{H}$ an element of the linear space $\mathcal{H}_L$ where N and L are the dimensions of the spaces. The mutually unique transformation ($L = N$) is carried out by the reversible operator, whereas the degenerate transformation ($L < N$) by an irreversible operator. The matrix of the operator h is called the *fundamental characteristic matrix*. This matrix can be *reduced* by deleting one or two rows.

For the sake of uniformity in presentation the formulas in parts I and II have common numeration.

Magnetic field and its parts

We shall construct a model consisting of an outer non-conducting domain $\mathcal{V}_e$ and inner conducting domain $\mathcal{V}_i$ containing local heterogeneities. The extrinsic current is distributed in the outer domain $\mathcal{V}_e$ or a primary field is given in it. The secondary field appears due to the electromagnetic induction in the inner domain $\mathcal{V}_i$. The primary and secondary fields may be regarded as external and internal fields. If the domain $\mathcal{V}_i$ is homogeneous, then the total field will be reckoned as a normal field. The local heterogeneities in the inner domain $\mathcal{V}_i$ create an anomalous field.

We shall confine ourselves to a consideration of the relationships between the components of the magnetic fields though such relationships can also be derived for the components of electrical fields as well.

Let $\vec{H}^t$ be the total magnetic field observed in our model. Decompose $\vec{H}^t$ into primary and secondary or normal and anomalous parts :

$$\vec{H}^t(r) = \begin{cases} \vec{H}^p(\vec{r}) + \vec{H}^s(\vec{r}) \\ \vec{H}^n(\vec{r}) + \vec{H}^a(\vec{r}) \end{cases}$$

where $\vec{H}^p$ is the primary field (i.e. the field that exists when there is no inner domain $\mathcal{V}_i$), $\vec{H}^n$ is the normal field (i.e. the field that exists when there are no local heterogeneities in the domain $\mathcal{V}_i$), $\vec{H}^s$ and $\vec{H}^a$ are the secondary and anomalous fields respectively. The methods and difficulties of such a decomposition are described in the literature (Chapman and Bartels, 1940 ; Rikitake, 1966 ; Schmucker, 1970, 1971 ; Berdichewski and Zhdanov, 1973, 1974 ; Zhdanov, 1973 ; Berdichewski, Zhdanov, Zhdanova, 1974).

The fields $\vec{H}^p$, $\vec{H}^s$, $\vec{H}^n$ and $\vec{H}^a$ form the linear spaces $\mathcal{H}^p_{L_p}$, $\mathcal{H}^s_{L_s}$, $\mathcal{H}^n_{L_n}$ and $\mathcal{H}^a_{L_a}$ whose dimensions, generally speaking, may differ from the dimension of the linear space $\mathcal{H}^t_{L_t}$. For instance, the primary magnetic field of an arbitrarily oriented electrical dipole has no radial component, and consequently the dimension of the space $\mathcal{H}^p_{L_p}$ does not exceed 2. At the same time the dimension L_s of the space $\mathcal{H}^s_{L_s}$ and consequently the dimension L_t of the space $\mathcal{H}^t_{L_t}$ can be equal to 3. The situations are also possible where the dimensions of the spaces $\mathcal{H}^p_{L_p}$ and $\mathcal{H}^s_{L_s}$ exceed the dimension of the space $\mathcal{H}^t_{L_t}$ (for example, on the surface of a perfect conductor).

We shall now introduce the characteristic operators h^t, h^p, h^s, h^n and h^a which transform the vector $\vec{M}$ into the vectors $\vec{H}^t$, $\vec{H}^p$, $\vec{H}^s$, $\vec{H}^n$ and $\vec{H}^a$ as follows :

$$\vec{H}^\alpha(\vec{r}) = h^\alpha(\vec{r})\vec{M} \qquad \alpha = t, p, s, n, a \qquad (20)$$

The operator h^α acts from the space $\mathfrak{M}_N$ into the space $\mathcal{H}^\alpha_{L_\alpha}$. If it is reversible then the dimensions of the spaces $\mathfrak{M}_N$ and $\mathcal{H}^\alpha_{L_\alpha}$ are the same ($L_\alpha = N$).

Induction operators

We shall take a magnetic field from some algebraic class and examine three cases.

1) At a given point $\vec{r}$ let all the operators h^β ($\beta = t, p, s, n, a$) be reversible. Then

$$\vec{M} = [h^\beta(\vec{r})]^{-1}\,\vec{H}^\beta(\vec{r}) \qquad (21)$$

(summation is not carried out with respect to β).

Substituting (21) into (20) we obtain linear relationships between any pairs of the magnetic fields under consideration :

$$\vec{H}^\alpha(\vec{r}) = \mathcal{J}^{\alpha\beta}(\vec{r})\,\vec{H}^\beta(\vec{r}) \qquad \alpha, \beta = t, p, s, n, a \quad (22)$$

where

$$\mathcal{J}^{\alpha\beta}(\vec{r}) = h^\alpha(\vec{r})\,[h^\beta(\vec{r})]^{-1} \qquad (23)$$

The linear operators $\mathcal{J}^{\alpha\beta}$ will be called the *induction operators* of the magnetic field. These operators transform the total magnetic field and its parts one into the other. *To each algebraic class there correspond its own operators $\mathfrak{J}^{\alpha\beta}$ which depend on the frequency, the observation site and the distribution of the electromagnetic properties of the medium.* Within a given algebraic class the induction operators can be regarded as the characteristics of the medium.

In the case considered above all the induction operators are reversible :

$$\mathfrak{J}^{\beta\alpha}(\vec{r}) = [\mathfrak{J}^{\alpha\beta}(\vec{r})]^{-1} \tag{24}$$

More often the geophysicists deal with the operator $\mathfrak{J}^{an}$ that interconnects the anomalous field with the normal field (Schmucker, 1970). We shall reckon this induction operator as the basic operator.

2) If some of the characteristic operators h^β ($\beta = t, p, s, n, a$) are irreversible, then the number of induction operators decreases. For example, if the operator h^t is irreversible, then the operators $\mathfrak{J}^{st}$ and $\mathfrak{J}^{at}$ vanish, whereas the operators $\mathfrak{J}^{tp}$ and $\mathfrak{J}^{tn}$ become irreversible. Consequently, the system of transformations (22) is partially decayed and the relationships (24) ceases to be valid.

3) If all the characteristic operators h^β ($\beta = t, p, s, n, a$) are irreversible, then the induction operators do not exist and the system of transformations (22) completely decays.

The conditions for the existence and the reversibility of the induction operators are tabulated below.

Table II

Conditions for the existence and the reversibility of the induction operators

h^β \ h^α	Reversible	Irreversible
Reversible	Reversible operators $\mathfrak{J}^{\alpha\beta}$ and $\mathfrak{J}^{\beta\alpha}$ exist	Irreversible operators $\mathfrak{J}^{\alpha\beta}$ exist
Irreversible	Irreversible operators $\mathfrak{J}^{\beta\alpha}$ exist	Induction operators do not exist

The matrices of the induction operators will be called the *fundamental induction matrices*. These matrices may be *reduced* by deleting one or two rows. The shape of the induction matrices, as in the case of the magnetotelluric matrices, depends on the dimension of the characteristic vector space. *The elements of the induction matrices are the electrical conductivity functions.*

Induction matrices in the class with the space $\mathfrak{M}_3$

We shall embed the spaces $\mathfrak{M}_3$ and $\mathcal{H}^\alpha_{L_\alpha}$, $\mathcal{H}^\beta_{L_\beta}$ (L_α, $L_\beta \leqslant 3$) into a three-dimensional physical space Φ_3. Expressing the fundamental matrices of the characteristic and induction operators in an arbitrary Cartesian basis $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ of this space we obtain the square matrices of order 3 :

$$h^\alpha_{ij},\ h^\beta_{ij},\ \mathfrak{J}^{\alpha\beta}_{ij} \qquad (i, j = 1, 2, 3).$$

By virtue of (23) we have

$$\mathfrak{J}^{\alpha\beta}_{ij}(\vec{r}) = h^\alpha_{ik}(\vec{r})\,[h^\beta_{kj}(\vec{r})]^{-1} \tag{25}$$

$$\alpha, \beta = t, p, s, n, a \qquad i, j, k = 1, 2, 3$$

The *induction tensors* acting in the space Φ_3 correspond to the fundamental induction matrices. According to (22) we have

$$H^\alpha_i(\vec{r}) = \mathfrak{J}^{\alpha\beta}_{ij}(\vec{r})\,H^\beta_j(\vec{r}) \tag{26}$$

$$\alpha, \beta = t, p, s, n, a \qquad i, j = 1, 2, 3$$

(summation is not carried out with respect to β).

The induction matrices $\mathfrak{J}^{\alpha\beta}$ exist if the characteristic operators h^β are reversible, i.e. the determinants of their matrices are non-zero. For example, let the operator h^n be reversible. Then the matrix $\mathfrak{J}^{an}$ exists and consequently any component of the field $\vec{H}^a$ can be expressed as a linear combination of three components of the field $\vec{H}^n$. The coefficients of this linear combination are the electrical conductivity functions.

The irreversibility of the characteristic operators reduce the number of linear combinations caused by the existence of the induction matrices $\mathfrak{J}^{\alpha\beta}$, but generates new linear combinations owing to the degeneracy of the characteristic matrices h^β.

Let the operator h^β ($\beta = t, p, s, n, a$) be irreversible, and the dimension of the space $\mathcal{H}^\beta_{L_\beta}$ be 2. Therefore, the degenerate matrix h^β_{ij} of rank 2 has at least one basic minor of order 2, for example, the minor

$$q^\beta_{33} = h^\beta_{11}\,h^\beta_{22} - h^\beta_{12}\,h^\beta_{21}$$

at the upper left-hand corner of the matrix (the first subscript indicates the number of the deleted row, while the second the number of the deleted column). According to the theorem on the basic minors, the third row of the matrix is a linear combination of the first and second rows :

$$h^\beta_{3j} = W^\beta_{31}\,h^\beta_{1j} + W^\beta_{32}\,h^\beta_{2j} \qquad j = 1, 2, 3$$

where

$$W^\beta_{31} = -\frac{q^\beta_{13}}{q^\beta_{33}} \qquad W^\beta_{32} = \frac{q^\beta_{23}}{q^\beta_{33}}$$

and

$$q^\beta_{13} = h^\beta_{21}\,h^\beta_{32} - h^\beta_{22}\,h^\beta_{31} \qquad q^\beta_{23} = h^\beta_{11}\,h^\beta_{32} - h^\beta_{12}\,h^\beta_{31}$$

Hence

$$H_3^\beta(\vec{r}) = h_{3j}^\beta(\vec{r}) M_j = W_{31}^\beta(\vec{r}) h_{1j}^\beta(\vec{r}) M_j + W_{32}^\beta(\vec{r}) h_{2j}^\beta(\vec{r}) M_j = W_{31}^\beta(\vec{r}) H_1^\beta(\vec{r}) + W_{32}^\beta(\vec{r}) H_2^\beta(\vec{r}), \quad (27)$$

i.e. the component H_3^β is the linear combination of linearily independent components H_1^β and H_2^β. These relationships are called the *Wiese-Parkinson relation* (Parkinson, 1959 ; Wiese, 1965). The matrices $W_{3\hat{j}}^\beta$ ($\hat{j} = 1, 2$) will be called the *Wiese-Parkinson matrices.*

The matrix $W_{3\hat{j}}^n$ reflects the polarization of the primary field. All other matrices W_{3j}^β ($\beta = t, s, n, a$) are associated with the polarization of the induced field and thus may be referred to the *induction matrices.* Their elements W_{31}^β and W_{32}^β are expressed in terms of the minors of the characteristic matrices, and consequently, they depend only on the frequency, the observation site and the distribution of the electromagnetic properties of the medium. *Within a given algebraic class the elements W_{31}^β and W_{32}^β can be regarded as the electrical conductivity functions.* The complex vector

$$\vec{W}^\beta = W_{31}^\beta \vec{d}_1 + W_{32}^\beta \vec{d}_2 \quad (28)$$

lying in the plane $x_1 x_2$ corresponds to them. This vector may be called the *Wiese-Parkinson vector* or the *induction vector* (induction arrow).

The matrices $W_{3\hat{j}}^\beta$ exist if the dimension L_β of the space $\mathcal{H}_{L_\beta}^\beta$ is 2. If $L_\beta = 1$, the matrix h_{ij}^β has no basic minors of order 2 and the Wiese-Parkinson relation for $h_{11}^\beta \neq 0$ and $h_{21}^\beta \neq 0$ reduces to the scalar relationship

$$H_3^\beta(\vec{r}) = W_{31}^\beta(\vec{r}) H_1^\beta(\vec{r}) = W_{32}(\vec{r}) H_2^\beta(\vec{r}) \quad (29)$$

where

$$W_{31}^\beta = \frac{h_{31}^\beta}{h_{11}^\beta} \qquad W_{32}^\beta = \frac{h_{31}^\beta}{h_{21}^\beta}$$

We shall give an example where the matrices $\mathcal{T}_{ij}^{\alpha\beta}$ and $W_{3\hat{j}}^\beta$ exist simultaneously. Let the operator h^p be reversible, whereas h^t and h^a be irreversible (the dimension of the spaces $\mathcal{H}_{L_t}^t$ and $\mathcal{H}_{L_a}^a$ is 2). Consequently, the fundamental induction matrices $\mathcal{T}_{ij}^{tp}$ and $\mathcal{T}_{ij}^{ap}$ exist, and therefore any component of the field $\vec{H}^t$ or $\vec{H}^a$ can be regarded as a linear combination of three components of the field $\vec{H}^p$. Moreover, the Wiese-Parkinson matrices $W_{3\hat{j}}^\beta$ and $W_{3\hat{j}}^\beta$ exist, therefore one of the components of the field $\vec{H}^t$ or $\vec{H}^a$ can be represented as a linear combination of two other components of the same field.

The fundamental induction matrices and the Wiese-Parkinson matrices can be uniquely determined. As an example consider the matrices $\mathcal{T}_{ij}^{an}$ and $W_{3\hat{j}}^t$ (the operator h^t is irreversible, and the space $\mathcal{H}_{L_t}^t$ is of dimension 2). Take three linearily independent vectors $\vec{H}^{n(1)}, \vec{H}^{n(2)}, \vec{H}^{n(3)}$ with the corresponding vectors $\vec{H}^{a(1)}, \vec{H}^{a(2)}, \vec{H}^{a(3)}$. Then by (26) we have

$$H_i^{a(k)}(\vec{r}) = \mathcal{T}_{ij}^{an}(\vec{r}) H_j^{n(k)}(\vec{r})$$

$$i, j, k = 1, 2, 3.$$

hence

$$\mathcal{T}_{ij}^{an}(\vec{r}) = H_i^{a(k)}(\vec{r}) [H_j^{n(k)}(\vec{r})]^{-1} \quad (30)$$

$$i, j, k = 1, 2, 3$$

where $[H_j^{n(k)}]^{-1}$ is the inverse of the matrix $H_j^{n(k)}$ formed by the components of three linearly indepedent vectors $\vec{H}^{n(k)}$.

For determining $W_{3\hat{j}}^t$ we shall take two vectors $\vec{H}^{t(1)}$ and $\vec{H}^{t(2)}$ with linearly independent components H_1^t and H_2^t. By virtue of (27) we have

$$H_3^{t(\hat{k})}(\vec{r}) = W_{3\hat{j}}^t(\vec{r}) H_{\hat{j}}^{t(\hat{k})}(\vec{r}) \quad \hat{j}, \hat{k} = 1, 2$$

hence

$$W_{3\hat{j}}^t(\vec{r}) = H_3^{t(\hat{k})}(\vec{r}) [H_{\hat{j}}^{t(\hat{k})}(\vec{r})]^{-1} \quad (31)$$

where $[H_{\hat{j}}^{t(\hat{k})}]^{-1}$ is the inverse of the matrix $H_{\hat{j}}^{t(\hat{k})}$ formed by the linearly independent components $H_1^{t(\hat{k})}$ and $H_2^{t(\hat{k})}$ of two vectors $\vec{H}^{t(\hat{k})}$.

The matrix $W_{3\hat{j}}^t$ is a special matrix among the Wiese-Parkinson matrices as it does not need the decomposition of the magnetic field into primary and secondary or normal and anomalous parts.

Induction matrices in the class with the space $\mathcal{M}_2$

Consider the characteristic operators h^α and h^β ($\alpha, \beta = t, p, s, n, a$) which act from the space $\mathcal{M}_2$ into the spaces $\mathcal{H}_{L_\alpha}^\alpha$ and $\mathcal{H}_{L_\beta}^\beta$ ($L_\alpha, L_\beta \leqslant 2$) transforming the vector $\vec{M}$ into the vectors $\vec{H}^\alpha$ and $\vec{H}^\beta$:

$$\vec{H}^\alpha(\vec{r}) = h^\alpha(\vec{r}) \vec{M} \quad \text{(a)}$$

$$\vec{H}^\beta(\vec{r}) = h^\beta(\vec{r}) \vec{M} \quad \text{(b)} \quad (32)$$

Embed the spaces $\mathcal{H}_{L_\alpha}^\alpha$ and $\mathcal{H}_{L_\beta}^\beta$ into a three-dimensional physical space $\mathcal{D}_3$. Introduce the basis $\{\vec{m}_1, \vec{m}_2\}$ for the space $\mathcal{M}_2$ and the Cartesian basis $\{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ for the space $\mathcal{D}_3$. In these bases,

$$H_i^\alpha(\vec{r}) = h_{i\hat{j}}^\alpha(\vec{r}) M_{\hat{j}} \qquad H_i^\beta(\vec{r}) = h_{i\hat{j}}^\beta(\vec{r}) M_{\hat{j}}$$

$$i = 1, 2, 3 \quad \hat{j} = 1, 2$$

The matrices $h^{\alpha}_{i\hat{j}}$ and $h^{\beta}_{i\hat{j}}$ are the fundamental characteristic matrices 3 x 2.

If the operator h^{β} is reversible, then by deleting some row from its rectangular matrix $h^{\beta}_{i\hat{j}}$ we can form at least one reduced square matrix with a non-zero determinant. We shall take the matrix $h^{\beta}_{\hat{i}\hat{j}}(\hat{i},\hat{j} = 1,2)$ as such a one. This matrix is formed by the first and the second rows of the rectangular matrix $h^{\beta}_{i\hat{j}}$. Its determinant is the basic minor q^{β}_3 of the fundamental matrix $h^{\beta}_{i\hat{j}}$. The reduced matrix $h^{\beta}_{\hat{i}\hat{j}}$ corresponds to the reversible operator $\tilde{h}^{\beta}$ which transforms the vector $\vec{M}$ into the vector $\vec{H}^{\beta}_{12}$:

$$\vec{H}^{\beta}_{12}(\vec{r}) = \tilde{h}^{\beta}(\vec{r})\vec{M}$$

Obviously, the vector $\vec{H}^{\beta}_{12}$ has linearly independent components and is the projection of the vector $\vec{H}^{\beta}$ on the plane $x_1 x_2$:

$$\vec{H}^{\beta}_{12}(\vec{r}) = H^{\beta}_{\hat{j}}(\vec{r})\vec{d}_{\hat{j}} \quad \hat{j} = 1,2$$

By virtue of the reversibility of the operator $\tilde{h}^{\beta}$ we have :

$$\vec{M} = [\tilde{h}^{\beta}(\vec{r})]^{-1}\,\vec{H}^{\beta}_{12}(\vec{r}) \tag{33}$$

(summation is not carried out with respect to β).

Substituting (33) into (32) we obtain

$$\vec{H}^{\alpha}(\vec{r}) = \mathfrak{J}^{\alpha\beta}(\vec{r})\,\vec{H}^{\beta}_{12}(\vec{r}) \quad \alpha,\beta = t,p,s,n,a \tag{34}$$

where

$$\mathfrak{J}^{\alpha\beta}(\vec{r}) = h^{\alpha}(\vec{r})\,[\tilde{h}^{\beta}(\vec{r})]^{-1} \tag{35}$$

In the Cartesian basis :

$$H^{\alpha}_{i}(\vec{r}) = \mathfrak{J}^{\alpha\beta}_{i\hat{j}}(\vec{r})\,H^{\beta}_{\hat{j}}(\vec{r}) \qquad i = 1,2,3 \quad \hat{j} = 1,2 \tag{36}$$

where

$$\mathfrak{J}^{\alpha\beta}_{i\hat{j}}(\vec{r}) = h^{\alpha}_{i\hat{k}}(\vec{r})\,[h^{\beta}_{\hat{k}\hat{j}}(\vec{r})]^{-1} \qquad i = 1,2,3 \quad \hat{j},\hat{k} = 1,2 \tag{37}$$

Thus we get the induction operators $\mathfrak{J}^{\alpha\beta}$ with the fundamental rectangular 3 x 2 matrices

$\mathfrak{J}^{\alpha\beta}_{i\hat{j}}(\alpha,\beta = t,p,s,n,a \;;\; i = 1,2,3 \;;\; \hat{j} = 1,2)$.

These operators act from the plane $x_1 x_2$ into a three-dimensional space and transform the projections of the field $\vec{H}^{\beta}$ into the field $\vec{H}^{\alpha}$. The third row of their matrices can be represented by the complex vector $\vec{S}$ lying in the plane $x_1 x_2$:

$$\vec{S}^{\alpha\beta} = \mathfrak{J}^{\alpha\beta}_{31}\vec{d}_1 + \mathfrak{J}^{\alpha\beta}_{32}\vec{d}_2 \tag{38}$$

This induction vector may be called the *Schmucker vector*. The basic Schmucker vector is $\vec{S}^{an}$.

The relationships (36) contain the Wiese-Parkinson relation. For $i = 3$ and $\alpha = \beta$ we have

$$H^{\beta}_3(\vec{r}) = W^{\beta}_{31}(\vec{r})H^{\beta}_1(\vec{r}) + W^{\beta}_{32}(\vec{r})H^{\beta}_2(\vec{r})$$

where

$$W^{\beta}_{31} = \mathfrak{J}^{\beta\beta}_{31} \quad W^{\beta}_{32} = \mathfrak{J}^{\beta\beta}_{32}$$

Thus we have the Wiese-Parkinson matrix $W^{\beta}_{3\hat{j}}(\hat{j} = 1,2)$ whose elements could be expressed in terms of the minors of the characteristic matrix $h^{\beta}_{i\hat{j}}$:

$$W^{\beta}_{31} = -\frac{q^{\beta}_1}{q^{\beta}_3} \quad W^{\beta}_{32} = \frac{q^{\beta}_2}{q^{\beta}_3}$$

where

$$q^{\beta}_1 = h^{\beta}_{21}h^{\beta}_{32} - h^{\beta}_{22}h^{\beta}_{31} \qquad q^{\beta}_2 = h^{\beta}_{11}h^{\beta}_{32} - h^{\beta}_{12}h^{\beta}_{31}$$

$$q^{\beta}_3 = h^{\beta}_{11}h^{\beta}_{22} - h^{\beta}_{12}h^{\beta}_{21}$$

Furthermore, we have the induction operators $\tilde{\mathfrak{J}}^{\alpha\beta}$ with the reduced square matrices $\mathfrak{J}^{\alpha\beta}_{\hat{i}\hat{j}}$ of order 2 $(\alpha,\beta = t,p,s,n,a \;;\hat{i},\hat{j} = 1,2)$. These operators act in the plane $x_1 x_2$ and transform the projections of the field $\vec{H}^{\beta}$ into the projections of the field $\vec{H}^{\alpha}$. The *induction tensors* acting in the same plane $x_1 x_2$ correspond to them :

$$H^{\alpha}_{\hat{i}}(\vec{r}) = \mathfrak{J}^{\alpha\beta}_{\hat{i}\hat{j}}(\vec{r})\,H^{\beta}_{\hat{j}}(\vec{r}) \qquad \hat{i},\hat{j} = 1,2 \tag{39}$$

The induction matrices are determined in the same manner as(30) or (31). For example

$$\mathfrak{J}^{\alpha n}_{i\hat{j}}(\vec{r}) = H^{\alpha(\hat{k})}_{i}(\vec{r})\,[H^{n(\hat{k})}_{\hat{j}}(\vec{r})]^{-1}$$

$$W^{t}_{3\hat{j}}(\vec{r}) = H^{t(\hat{k})}_{3}(\vec{r})\,[H^{t(\hat{k})}_{\hat{j}}(\vec{r})]^{-1} \tag{40}$$

$$i = 1,2,3 \qquad \hat{j},\hat{k} = 1,2$$

where $[H^{n(k)}_{\hat{j}}]^{-1}$ and $[H^{t(k)}_{\hat{j}}]^{-1}$ are the inverse of the matrices $H^{n(k)}_{\hat{j}}$ and $H^{t(k)}_{\hat{j}}$ which are formed by the linearly independent components H_1 and H_2 of two vectors $\vec{H}^{n}$ or $\vec{H}^{t}$.

Thus, the induction matrices $\mathfrak{J}^{\alpha\beta}_{i\hat{j}}$ and $W^{\beta}_{3\hat{j}}$ exist if the characteristic operators h^{β} are reversible, i.e. if the rank of their matrices is 2. Each of the rectangular matrices $\mathfrak{J}^{\alpha\beta}_{i\hat{j}}$ can be reduced to a tensor matrix of order 2. For instance, let the operator h^{n} be reversible. Then the matrices $\mathfrak{J}^{\alpha n}_{i\hat{j}}$ and $W^{n}_{3\hat{j}}$ exist causing the existence of appropriate linear combinations.

The irreversibility of the characteristic operators diminishes the number of linear combinations. For instance, if h^n is irreversible, the linear combinations associated with the existence of the matrices $\mathcal{J}_{i\hat{j}}^{nn}$ vanish, whereas the Wiese-Parkinson relation reduces to a linear dependence of any two components of the field $\vec{H}^n$.

Induction matrices in the class with the space $\mathfrak{M}_1$

The fundamental induction matrices are column matrices. On reducing the fundamental matrices, we obtain the scalar proportionality coefficients :

$$H_i^\alpha(\vec{r}) = \mathcal{J}_{ij}^{\alpha\beta}(\vec{r})\, H_j^\beta(\vec{r}) \qquad i, j = 1, 2, 3 \tag{41}$$

where the summation is not carried out with respect to j, β. Any two Cartesian components of the fields $\vec{H}^\alpha$ and $\vec{H}^\beta$ are linearly dependent. If $\alpha = \beta$ then this dependence gives the Wiese-Parkinson relation.

Main geophysical applications

Fields with different excitation mechanisms and different spatial configurations are generally studied in the magnetotelluric and magnetovariation methods. These fields can be classified into four basic types : a) *pulsations*, b) *polar substorms (bays)*, c) *quiet solar diurnal variations*, d) *world storms*. We shall show that all these fields can be reduced to the model fields of the algebraic classes discussed in our paper (Part I). These model fields are tabulated below.

The magnetotelluric and induction matrices will be considered in points on the Earth's surface with a local basis consisting of two horizontal unit vectors $\vec{d}_1$, $\vec{d}_2$ and a vertical unit vector $\vec{d}_3$.

The Earth is reckoned heterogeneous. Since the atmosphere has very low electrical conductivity, we shall assume that the field $\vec{E}$ has no vertical component on the inner side of the Earth's surface (Eckhardt et al. 1963 ; Swift, 1967 ; Berdichewski, Dmitriev, Van'yan, 1971), i.e. the dimension of the space $\mathcal{E}_K$ does not exceed 2. Simple models show that in this case there are no such restrictions for the magnetic field. Therefore we shall suppose that the space $\mathcal{H}_L$ has the same dimension as the space $\mathfrak{M}_N$ except in special situations.

In this paper we shall confine ourselves only to a brief remarks on the nature of the investigated fields, their models, the shape of the magnetotelluric and induction matrices.

Pulsations. Today it is believed that the pulsations are developed as a result of the transformation of Alfven waves or magneto-sound waves generated in the magnetosphere (Van'yan, Abramov et al. 1973 ; Gokhberg et al. 1973). In the middle and low latitudes the pulsations are approximated by the field of an arbitrarily oriented horizontal electrical dipole localized in the polar ionosphere (Berdichewski, Van'yan, Osipova, 1972, 1973), or by an arbitrarily polarized non-uniform plane wave consisting of TE and TM modes (Madden, Swift, 1969). By neglecting the displacement of the ionospheric dipole or the variations in the wave vector we obtain the model I or IV. Following Madden and Swift, we shall eliminate the conduction mode TM from model IV and thus pass on to the model with the induction mode TE which if needed can be reduced to the Cagniard's model with uniform plane wave falling vertically (Cagniard, 1953). The Cagniard's model is very convenient in practice since it does not require any information about the field source geometry and gives the electrical conductivity functions depending only on the frequency, the observation site and the properties of the medium. Its shortcoming is that the vertical component of the primary magnetic field is neglected. All these model fields form the algebraic class with the space $\mathfrak{M}_2$. Thus, for reversible operators e and h, and linearily independent horizontal components of the field, the fundamental magnetotelluric and induction matrices 3 x 2 are $Z_{i\hat{j}}$, $Y_{i\hat{j}}$, $t_{i\hat{j}}$, $m_{i\hat{j}}$ and $\mathcal{J}_{i\hat{j}}^{\alpha\beta}$ where $i = 1, 2, 3$; $\hat{j} = 1, 2$; $\alpha, \beta = t, p, s, n, a$ and $Z_{31} = Z_{32} = 0$, $t_{31} = t_{32} = 0$. The Wiese-Parkinson matrix is $W_{3\hat{j}}^\beta$ ($\hat{j} = 1, 2$; $\beta = t, s, n, a$) where $W_{31}^\beta = \mathcal{J}_{31}^{\beta\beta}$, $W_{32}^\beta = \mathcal{J}_{32}^{\beta\beta}$. The fundamental magnetotelluric and induction matrices are reduced to tensor matrices $Z_{\hat{i}\hat{j}}$, $Y_{\hat{i}\hat{j}}$, $t_{\hat{i}\hat{j}}$, $m_{\hat{i}\hat{j}}$ and $\mathcal{J}_{\hat{i}\hat{j}}^{\alpha\beta}$ ($\hat{i}, \hat{j} = 1, 2$; $\alpha, \beta = t, p, s, n, a$) acting in the horizontal plane.

It would be interesting to study some special cases, for instance, the linear polarization of the field $\vec{E}$ caused by the action of enlongated non-conducting structure (Berdichewski et al. 1970). In this case the operator e is irreversible and the matrices $Y_{i\hat{j}}$, $t_{i\hat{j}}$ do not exist.

Another example is associated with the action of enlongated conducting structures (Schmucker, 1970). This case is characterized by the linear polarization of the anamolous field $\vec{H}^a$. Therefore the operator h^a is irreversible and the matrices $\mathcal{J}_{i\hat{j}}^{\alpha a}$, $W_{3\hat{j}}^a$ do not exist.

Polar substorms. Polar substorms are manifested in the form of baylike perturbations caused by the electrojet in the auroral zone (Isaev, Pudovkin, 1972). The drift of the electrojet along the auroral zone provokes a complex geometry of the primary field. Directly under the electrojet the bays can be simulated by an infinite rectilinear ionospheric current

Table 3

Model and Simulated Fields

Model	Characteristic vector $\vec{M}$	Dimension of space $\mathfrak{M}_N$	Simulated field
I. Motionless electrical dipole with horizontal moment $\vec{P}$	$\vec{M} = \vec{P}$	$N = 2$	Pulsations in the middle and low latitudes
II. Electrical or magnetic dipole with moment $\vec{P}(t)$ moving along a given rectilinear path with a given velocity and a given law for relative variation of moment :	$\vec{M} = \vec{P}(t_0)$		
a) horizontal dipole moment		$N = 2$	Bays in the middle and low latitudes
b) arbitrarily oriented dipole moment		$N = 3$	Sudden commencement of world magnetic storms
III. Infinitely long rectilinear current $\mathcal{J}$ flowing along a given line in the direction of a unit vector $\vec{d}_1$	$\vec{M} = \mathcal{J}\vec{d}_1$	$N = 1$	Bays in high latitudes
IV. Arbitrarily polarized non-uniform or uniform plane wave $\vec{H}^p = \vec{H}_0\, e^{-\vec{k}.\vec{r}}$ $\vec{E}^p = \vec{E}_0\, e^{-\vec{k}.\vec{r}}$ with a given wave vector $\vec{k}$	$\vec{M} = \vec{H}_0$	$N = 2$	Pulsations and bays in the middle and low latitudes
V. Motionless current eddy of constant configuration with a total magnetic moment $\vec{I}$	$\vec{M} = \vec{I}$	$N = 1$	Quiet solar diurnal variations in the middle latitudes
VI. Arbitrarily oriented uniform magnetic field $\vec{H}_0$	$\vec{M} = \vec{H}_0$	$N = 3$	Main phase of the world magnetic storm

(Van'yan, 1965), and it corresponds to the model III which generates the algebraic class with the space $\mathfrak{M}_1$. In this model any two components of $\vec{E}$ and $\vec{H}$ are linearly dependent according to (19) and (41). Evidently, the accuracy of such an approximation decreases in moving off from the electrojet.

In the middle and low latitudes the bays can be approximated by a plane wave falling vertically (model IV). Besides, by neglecting the various forms of bays, curvature of the auroral zone and the variations in the electrojet drift velocity we can take the model with horizontal electrical dipole moving along a rectilinear path (model IIa). Both these two models generate the algebraic class with the space $\mathfrak{M}_2$. Consequently, in the middle and low latitudes the transition from the pulsations to bays does not change the structure of the magnetotelluric and induction matrices.

Quiet solar diurnal variations. These variations are caused by the Earth's rotation in the magnetic field of the ionospheric current eddies (Chapman, Bartels, 1940). The eddy centres are localized in latitudes of about 30°. By neglecting the changes of eddy geometry and using a laboratory reference frame we can represent the ionospheric current as a set of motionless monoharmonic eddies of constant configuration (model V). In such a model the time harmonics of the quiet solar diurnal variations form the algebraic class with the space $\mathfrak{M}_1$ which gives rise to a linear dependence of any two components of $\vec{E}$ and $\vec{H}$. These relationships are found to be valid in the middle latitudes (40-60°) where the changes of the ionospheric current geometry have hardly any influence.

World magnetic storms. We shall confine ourselves to a consideration of two phases of the world magnetic storms, namely, the sudden commencement and the main phase.

The sudden commencement of magnetic stroms is generally simulated by the motion of a fictitious magnetic dipole obtained as a result of reflection of the main dipole field of the Earth at the deformed surface of the magnetosphere (Ferraro, 1952). A dipole at a distance of 10-20 radii from the Earth moves from the Sun towards the Earth and on the average is normal to the magnetic equatorial plane. From one storm to the other the orientation of the dipole may vary in any direction within the limits of some tens of degrees. Neglecting the different durations of the process and the variations in the dipole trajectory and velocity we obtain the model 11b. Therefore, we can refer the sudden commencement of magnetic storms to the algebraic class with the space $\mathfrak{M}_3$.

The main phase of the magnetic storm is developed as a result of the formation of ring current flowing in a plane close to the equatorial one and its radius is about five Earth's radii (Ben'kova, 1952). The orientation of the ring current plane may vary from storm to storm within the limits of some tens of degrees. Thus, the main phase of the magnetic storm is satisfactorily approximated by an arbitrarily oriented uniform magnetic field (model VI), and consequently, it belongs to the algebraic class with the space $\mathfrak{M}_3$.

As we see, both these phases of the magnetic storm generate the algebraic class with the space $\mathfrak{M}_3$. In this class the operator e is always irreversible as $E_3 = 0$. Consequently, when the operator h is reversible, the following fundamental square matrices of order 3 exist :

$$Z_{ij}\,,\, m_{ij}\,,\, \mathfrak{J}^{\alpha\beta}_{ij} \quad (i\,,j = 1,2,3 \;;\; \alpha\,,\beta = t,\,p,\,s,\,n,\,a)$$

where $Z_{31} = Z_{32} = Z_{33} = 0$. The impedance, magnetic and induction tensors acting in the three-dimensional space correspond to them.

We must emphasize here that *only for the world magnetic storms it is possible to determine the magnetotelluric and induction matrices of order 3* (provided the changes in the polarization of the magnetic field are sufficiently strong). In this case the Wiese-Parkinson relations may exist only under certain special conditions. For instance, if the anomalous field $\vec{H}^a$ caused by conducting structure is of the conductive type and weakly associated with the vertical component of the field $\vec{H}^n$ (Rokityanski, 1972), then we have

$$H^a_i(\vec{r}) \approx \mathfrak{J}^{an}_{i\hat{j}}(\vec{r})\, H^n_{\hat{j}}(\vec{r}) \qquad \begin{matrix} i = 1,2,3 \\ \hat{j} = 1,2 \end{matrix}$$

hence

$$H^a_3(\vec{r}) \approx W^a_{3\hat{j}}(\vec{r})\, H^a_{\hat{j}}(\vec{r}) \qquad \hat{j} = 1,2$$

where

$$W^a_{31} = \frac{\mathfrak{J}^{an}_{31}\mathfrak{J}^{an}_{22} - \mathfrak{J}^{an}_{21}\mathfrak{J}^{an}_{32}}{\mathfrak{J}^{an}_{11}\mathfrak{J}^{an}_{22} - \mathfrak{J}^{an}_{12}\mathfrak{J}^{an}_{21}} \qquad W^a_{32} = \frac{\mathfrak{J}^{an}_{11}\mathfrak{J}^{an}_{32} - \mathfrak{J}^{an}_{12}\mathfrak{J}^{an}_{31}}{\mathfrak{J}^{an}_{11}\mathfrak{J}^{an}_{22} - \mathfrak{J}^{an}_{12}\mathfrak{J}^{an}_{21}}$$

A second example is also very interesting. Let us assume that the anomalous field H^a_3 is much stronger than the normal one : $|H^a_3| \gg |H^n_3|$ i.e. $H^a_3 \approx H^t_3$. Then from the relationship

$$H^a_3(\vec{r}) = \mathfrak{J}^{at}_{3j}(\vec{r})\, H^t_j(\vec{r}) \qquad j = 1,2,3$$

for $\mathfrak{J}^{at}_{33} \neq 1$ we obtain

$$H^t_3(\vec{r}) \approx W^t_{3\hat{j}}(\vec{r})\, H^t_{\hat{j}}(\vec{r}) \qquad \hat{j} = 1,2$$

where

$$W^t_{31} = \frac{\mathfrak{J}^{at}_{31}}{1 - \mathfrak{J}^{at}_{33}} \qquad W^t_{32} = \frac{\mathfrak{J}^{at}_{32}}{1 - \mathfrak{J}^{at}_{33}}$$

Magnetic field on the Moon. The interplanetary magnetic field is almost uniform at distances of the order of one Moon's diameter. Its orientation may vary over rather a wide range (Sonett et al. 1974). Thus, we obtain the model VI with an arbitrarily oriented primary uniform magnetic field $\vec{H}^p$ filling the Moon and its cavity (Van'yan et al. 1973). This model generates the algebraic class with the space $\mathfrak{M}_3$. The secondary field $\vec{H}^s$ is generated as a result of induction in the Moon.

On the day side subjected to the conducting solar wind we have $H^s_3 = 0$ and $H^a_3 = 0$. Consequently, the operators h^s and h^a are irreversible which restricts the choice of the induction matrices. In fact, only the following matrices may exist :

$$\mathfrak{J}^{\alpha\beta}_{ij} \quad (i\,,j = 1,2,3 \;;\; \alpha = t,\,p,\,s,\,n,\,a \;;\; \beta = t,\,p,\,n)$$

where $\mathfrak{J}^{\alpha\beta}_{31} = \mathfrak{J}^{\alpha\beta}_{32} = \mathfrak{J}^{\alpha\beta}_{33} = 0$ if $\alpha = s,\,a$. The induction tensors acting in the three-dimensional space correspond to these matrices.

There are no such restrictions for the dark side of the Moon.

Discussion

An example of empirical approach may be found in the monograph by Rokityanski (1972). The author establishes the shape of the magnetotelluric and induction matrices irrespective of the variation type and introduces the Wiese-Parkinson relation voluntarily. These short-comings clearly demonstrate the difficulties that arise because of neglecting the functional nature of the linear relationships between the components of the electromagnetic field.

The paper by Lilley and Bennet (1972) belongs to the most interesting ones. The Wiese-Parkinson relation is discussed in detail in it. The authors believe that only the degeneration of the induction matrix of order 3 (i.e. the irreversibility of the operator h in the algebraic class with the space $\mathfrak{M}_3$) is responsible for the appearance of the Wiese-Parkinson relation. The models which we examined here do not refute such a possibility, but we think that it is more of an exception than of a rule. Lilley and Bennet reject the second cause associated with the linear dependence of the primary field components (i.e. with the transition to the algebraic class with the space $\mathfrak{M}_2$). Assuming the existence of the matrix $W^p_{3\hat{f}}$ ($\hat{f} = 1, 2$) they come to conclusion "that a well-established Parkinson vector, formed from an ensemble of events by a consistent correlation of H^p_3 with some horizontal component of the primary field would require a consistent H^p_1/H^p_2 ratio" which contradicts the structure of the primary field because "there is no consistent repeatibility of the ionospheric currents". But this reasoning seems to be wrong since the good determination of a Parkinson vector testifies to consistent correlation between H^t_3 and some horizontal component of the total field, but not between H^p_3 and some horizontal component of the primary field. Our models, for example, the model of a heterogeneous medium excited by plane wave show that just the linear dependence of the primary field components is the main cause of the Wiese-Parkinson relation (at least for bays and pulsations).

Conclusions

We can thus assume that the linear relationships between the components of the Earth's (or the Moon's) electromagnetic field are functional rather than stochastic in nature. If it is true, the magnetotelluric process will consist of two parts : a) the main part corresponding to the model field of the algebraic class, this part is an ideal linear system with constant parameters (electrical conductivity functions), b) noises arising due to the discrepancies between the real and model fields. Hence, the statistical calculation of the electrical conductivity functions reduces to spectral analysis of the field, and the determination of its linear part (an ideal system) corresponding to the model field of the algebraic class. The structure of the ideal linear system (hence, the shape of the magnetotelluric and induction matrices) may be established a priory by the type of the variations and dimension of the characteristic vector space of the proposed model field. If the errors of the measurement and spectral analysis are negligibly small, the deviation of the multiple coherence function from unity (or the normalized residual of the multiple regression from zero) characterizes the degree of the discrepancies between the real and model fields.

We believe that such an approach gives a clear idea of the problem, at least, in studying the basic types of electromagnetic variations.

In what measure is the functional nature of the linear relationships between Earth's electromagnetic field components verified in practice ? This should be the matter of special article, and we can cite here only some typical facts :

– according to Porstendorfer (1961), Berdichewski (1961, 1965), Keller and Frischknecht (1966), the scatter of points on relative ellipses applied for the determination of the telluric and magnetic matrices in the telluric current method and by the magnetovariation profiling is rather small (5 – 15 %).

– according to Berdichewski (1968), Vladimirov and Krylov (1969), the scatter of elements of the impedance matrix hand-calculated from individual quasi-sinusoidal pulsations is rather small, too (10-15 % in modulus).

– according to Berdichewski, Bezruk, Chinareva (1973) and Berdichewski, Kolmanski, Ozerov (1974), the scatter of elements of the impedance matrix obtained from short series of pulsations by mathematical filtration or by disclosure of hidden periodicity is as a rule small (5-10 % in modulus).

– according to Reddy and Rankin (1974), the multiple coherence function exceeds 0.9 - 0.95 in the range of periods from 35 to 2200 sec ; therefore the linear part of the field predominates over the nonlinear.

All this corroborates the validity of our assumptions.

The results obtained for the induction matrices are less convincing. The points on the Wiese graphs are greatly scattered (50 % and even more) and the normalized residuals reach 0.5 - 0.7 (Rokityanski, 1972 ; Schmucker, 1970). But what is the reason of such poor correlation ? Is it the slight functional connection, or non-adequacy of the graphical methods and errors of spectral analysis ? The question seems not to be clear.

We understand that our hypotheses about the structure of the external field are schematical and some episodic breaking of the functional relation between the field components may occur, especialy when dealing with pulsations and bays in high latitudes. But we would like to finish our paper with the following optimistic suggestion : *let's seek the reason of instability of the magnetotelluric and induction matrices first of all in the imperfection of the analysis technique because only in this way we shall be able to improve the accuracy of the magnetotelluric and magnetovariation methods.*

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