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Use of cauchy integral analogs in the geopotential field theory

by

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ABSTRACT. – *A new technique of vector Cauchy integral analogs has been developed for three-dimensional potential fields which extends the basic principles of the classical theory of Cauchy integrals to three-dimensional cases. Representations have been derived for gravitational and magnetic fields of three-dimensional perturbing bodies with arbitrary density or magnetization distribution in the form of certain vector Cauchy integral analogs over the surface of bodies. Several important questions in the theory of analytical continuation of three-dimensional geopotential fields inside masses and the interrelations between the singular points and the geometry of the surface of perturbing bodies and the density or magnetization distribution inside these bodies have been investigated with the help of these representations. Thus, it has been shown that the basic results of the two-dimensional geopotential field theory can be extended to three-dimensional cases with the help of the Cauchy integral analog technique.*

RESUME. – *On développe une nouvelle technique d'analogues vectoriels à l'intégrale de Cauchy pour des champs potentiels tridimensionnels, qui étend les principes de base de la théorie classique des intégrales de Cauchy aux cas tridimensionnels. On dérive des représentations pour les champs gravitationnel et magnétique de corps perturbateurs tridimensionnels de densité ou d'aimantation quelconques sous la forme de certains analogues vectoriels de l'intégrale de Cauchy sur la surface des corps. On étudie par ces représentations diverses questions importantes de la théorie du prolongement analytique des champs géopotentiels tridimensionnels à l'intérieur des masses et les relations entre les points singuliers, la géométrie de la surface des corps perturbateurs et la distribution de densité ou l'aimantation à l'intérieur de ces corps. On a ainsi montré que les résultats de base de la théorie du champ géopotnetiel bidimensionnel peuvent être étendus aux cas tridimensionnels grâce à la technique de l'analogue de l'intégrale de Cauchy.*

Introduction

Most of the advances made during the last twenty years in the interpretation of two-dimensional potential fields have been achieved due to the use of the techniques of the theory of functions of a complex variable. This approach has been developed in detail by V.N. Strakhov, G.M. Voskoboinikov, G.Ya. Golizdra, A.V. Tsirulskiy and many others.

Cauchy type integrals play an exceptionally important part in the theory of logarithmic potential (as in the theory of functions of a complex variable). They are used in the analytical continuation of fields, in finding the location and properties of their singular points and in determining the uniqueness of the solution of the inverse problem.

The importance and the need for generalizing the results of the two-dimensional theory to three-dimensional case are quite evident. These problems were investigated first by V.N. Strakhov (Strakhov, 1970 ; 1974) who also studied an important particular case of axisymmetric problem (Strakhov, 1976).

In the papers (Zhdanov, 1973 ; 1974 ; 1975 ; 1976) it has been shown that several results of the logarithmic potential theory can be extended to three-dimensional case, using certain analogs of the Cauchy type integrals for the three-dimensional fields which are the modifications of the integrals introduced by Moisil, Teodoresko and Bitsadze (Bitsadze, 1953 ; 1972(**). In the present paper a new approach, in our opinion, much simpler than that used in the Moisil-Teodoresko-Bitsadze theory,

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(**) A similar approach to solve this problem has also been used by T. Kolbenheyer (1976 ; 1978), E. Vargova (1977), and L. Sitarova (1977) who investigated the three-dimensional potential problems, using the four-vectors analytical in the sense of Moisil and Teodoresko.

is proposed for constructing the three-dimensional analogs of Cauchy integrals. The three-dimensional problems of gravimetry and magnetometry both in the case of uniformly perturbing bodies as well as in the case of arbitrary density or magnetization distribution have been studied with the help of this technique. A striking feature of the relationships thus obtained is that in the case of two-dimensional fields these expressions readily reduce to the corresponding expressions of the complex theory of logarithmic potential. This feature of our approach to solve the three-dimensional problems of the potential theory is quite essentiel and useful in generalizing the techniques of the theory of functions of a complex variable to three-dimensional cases. Note that a similar technique can be developed for transient electromagnetic fields as well. Some results of such a generalizations were reported at the 1V Workshop on Electromagnetic Induction in the Earth and Moon held at Murnau (Zhdanov, 1980).

1. Three-dimensional Analogs of Cauchy Integrals

First we shall recall how the Cauchy type integral is defined in the theory of functions of a complex variable.

The Cauchy integral theorem is formulated as follows : If $f(\zeta)$ is a function analytical inside a domain D bounded by a contour L and continuous in $\bar{D}$ (where $\bar{D}$ is the closure of D), then from the values of $f(\zeta)$ on L, its value anywhere inside D can be determined by means of the expression :

$$f(\zeta') = \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\zeta - \zeta'} d\zeta \; ; \quad \zeta' \in D \qquad (1a)$$

(The orientation of the contour L is such that the domain D remains on the left).

If the point ζ' lies outside $\bar{D}$, then

$$\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\zeta - \zeta'} d\zeta = 0 \; ; \quad \zeta' \in C\bar{D} \qquad (1b)$$

(where $C\bar{D}$ is the complement of $\bar{D}$ with respect to the whole space).

Note that the Cauchy integral (1) exists also in the case where an arbitrarily piecewise smooth closed or open contour L is given on the complex plane and an arbitrary continuous function $\varphi(\zeta)$ is defined on L :

$$K(\zeta') = \frac{1}{2\pi i} \int_L \frac{\varphi(\zeta)\, d\zeta}{\zeta - \zeta'} \qquad (2)$$

Integral (2) is called *Cauchy type integral*, and the function $\varphi(\zeta)$, its *density*. Cauchy type integral has certain remarkable properties, of which the main are (Bitsadze, 1972) :

1) The function K defined by the Cauchy type integral is analytical everywhere outside the contour L;

2) If L is a closed contour bounding a domain D, and the function φ is the limit on L of some function analytical in D, then

$$K(\zeta') = \begin{cases} \varphi(\zeta') \; ; & \zeta' \in D \; ; \\ 0 \; ; & \zeta' \in C\bar{D} \; ; \end{cases} \qquad (3)$$

3) If the density of Cauchy type integral satisfies the Holder boundary condition on L, then the function K has both left-hand and right-hand limits, when ζ' tends to any point ζ_0 belonging to the smooth part of the contour L and the limiting values are found from the Sokhotskiy-Plemel formulas :

$$K^+(\zeta_0) = \lim_{\zeta' \to \zeta_0} K(\zeta') = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)\, d\zeta}{\zeta - \zeta_0} + \frac{1}{2}\varphi(\zeta_0) \; ; \quad \zeta' \in D \; ; \qquad (4)$$

$$K^-(\zeta_0) = \lim_{\zeta' \to \zeta_0} K(\zeta') = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)\, d\zeta}{\zeta - \zeta} - \frac{1}{2}\varphi(\zeta_0) \; ; \quad \zeta' \in C\bar{D} \; ; \qquad (4)$$

We shall now show that a theory somowhat similar to the theory of Cauchy type integral can be developed for three-dimensional fields as well.

First we shall derive the three-dimensional analogs of the Cauchy integral formula. Let S denote a piecewise smooth surface bouding a domain D in a three-dimensional space. If $\mathbf{A}$ is a vector function continuously differentiable every-where on D, then

$$\int_D \iint \operatorname{div} \mathbf{A}\, dv = \int_S \int (\mathbf{A}, n)\, dS \qquad (5)$$

where $\mathbf{n}$ is a unit vector along the outer normal to S.

Suppose that the vector-function $\mathbf{A}$ is expressed in the form :

$$\mathbf{A} = (\mathbf{C}, \nabla h)\, \nabla f + [\nabla f \times [\nabla h \times \mathbf{C}]] \,, \qquad (6)$$

where h, φ are arbitrary functions twice continuously differentiable in $\bar{D}$, and $\mathbf{C}$ is an arbitrary constant vector. Substituting (6) in (5), and since the choice of $\mathbf{C}$ is arbitrary, we obtain the Gauss theorem in a vectorial form :

$$\iint_D \int (\Delta f \nabla h + \Delta h \nabla f)\, dV$$

$$= \int_S \int \{(\mathbf{n}, \nabla f)\, \nabla h + [[\mathbf{n} \times \nabla f] \times \nabla h]\}\, dS =$$

$$= \int_S \int \{(\mathbf{n}, \nabla h)\, \nabla f + [[\mathbf{n} \times \nabla h] \times \nabla f]\} dS \,. \qquad (7)$$

In (6) assume that h is a function harmonic in D, and $f = g(\mathbf{r} - \mathbf{r}')$ – is the Green function for the Laplace equation :

$$\Delta h \equiv 0 ; \quad \mathbf{r} \in D$$

$$\Delta f = \Delta g = -4\pi\delta(\mathbf{r} - \mathbf{r}') \tag{8}$$

$$g(\mathbf{r} - \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} ;$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function.

Hence

$$-\frac{1}{4\pi}\iint_S \left\{ (\mathbf{n}, \nabla h) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} + [\mathbf{n} \times \nabla h] \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|}\right] \Big\} dS = \begin{cases} \nabla h(\mathbf{r}') ; & \mathbf{r}' \in D \\ 0 ; & \mathbf{r}' \in C\bar{D} \end{cases} \tag{9}$$

Differentiation and integration on the left-hand side of (9) are carried out with respect to $\mathbf{r}$, while $\mathbf{r}'$ is taken to be a fixed point. In structure this expression is quite similar to the Cauchy integral formula. Indeed, just like the later, it is useful in reconstructing a field inside D from its values on the boundary, while the integral vanished outside D. Moreover, the apparent exterior resemblance reflects the internal unity of these expressions, because Eq. (9) reduces to (1) in two-dimensional case. It can, therefore, be called the analog of Cauchy formula. Proof of this fact is given in appendix A.

Now, using the expression (9), we shall construct the three-dimensional analog of the Cauchy type integral. First, we shall recall how the two-dimensional differentiation operators of fields on a surface S are defined :

$$\nabla^S h = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C v h \, dl ;$$

$$(\nabla^S , h_\tau) = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_c (h_\tau , v) \, dl ; \tag{10}$$

$$[\nabla^S \times h_\tau] = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{n}(h_\tau , \mathbf{l}) \, dl ;$$

where h is a scalar field, $\mathbf{h}_\tau$ is a vector field tangential to S, C is a contour bounding a part ΔS of the surface S, $\mathbf{n}$ is a unit vector normal to S, l-is a unit vector of the extrernal normal to C, belonging to S, l-is a unit vector along the tangent to C, and the vectors $\mathbf{n}$, v, $\mathbf{l}$ form a right-handed triplet, i.e $[\mathbf{n} \times v] = \mathbf{l}$. Now, on S define a continuously differentiable vector field $\varphi = \varphi_n + \varphi_\tau$ satisfying the condition :

$$[\nabla^S \times \varphi_\tau] \equiv 0 \tag{11}$$

where φ_τ and φ_n are the tangential and normal components of φ, respectively.

Write the following expression :

$$\mathbf{K}^S(\mathbf{r}', \varphi) = -\frac{1}{4\pi}\iint_S \left\{ (\mathbf{n}, \varphi) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} + \left[[\mathbf{n} \times \varphi] \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] \right\} dS . \tag{12}$$

Evidently, (12) exists at all points of the space not belonging to S.

We shall transform (12), using the obvious equality :

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} ,$$

(where ∇' shows an operating with respect to $\mathbf{r}'$) :

$$\mathbf{K}^S(\mathbf{r}', \varphi) = \frac{1}{4\pi}\nabla' \iint_S (\mathbf{n}, \varphi_n) \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS - \frac{1}{4\pi}[\nabla' \times \iint_S [\mathbf{n} \times \varphi_\tau] \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS] \tag{13}$$

This expression can be used to derive the equation verified by the vector field $\mathbf{K}^S$. Indeed, the divergence of $\mathbf{K}^S$ is

$$(\nabla', \mathbf{K}^S) = \frac{1}{4\pi}\iint_S (\mathbf{n}, \varphi_n) \Delta' \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS \equiv 0 \tag{14}$$

$$\mathbf{r}' \notin S$$

(where ∇', Δ' operates with respect to $\mathbf{r}'$).

We shall now calculate $[\nabla' \times \mathbf{K}^S]$:

$$\left[\nabla' \times \mathbf{K}^S\right] = -\frac{1}{4\pi}[\nabla' \times [\nabla' \times \iint_S [\mathbf{n} \times \varphi_\tau] \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS]\Big]$$

$$= -\frac{1}{4\pi}\nabla' \iint_S \left(\nabla' , \frac{[\mathbf{n} \times \varphi_\tau]}{|\mathbf{r} - \mathbf{r}'|}\right) dS \tag{15}$$

$$\mathbf{r}' \notin S$$

And we have

$$\left(\nabla' , \frac{[\mathbf{n} \times \varphi_\tau]}{|\mathbf{r} - \mathbf{r}'|}\right) = \left(\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} , [\mathbf{n} \times \varphi_\tau]\right) =$$

$$= -\left(\nabla^S \frac{1}{|\mathbf{r} - \mathbf{r}'|} , [\mathbf{n} \times \varphi_\tau]\right)$$

$$= \frac{(\nabla^S , [\mathbf{n} \times \varphi_\tau])}{|\mathbf{r} - \mathbf{r}'|} - \left(\nabla^S , \left[\mathbf{n} \times \frac{\varphi_\tau}{|\mathbf{r} - \mathbf{r}'|}\right)\right] \tag{16}$$

By virtue of (10) and (11)

$$(\nabla^S, [\mathbf{n}\times\varphi_\tau]) = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_C([\mathbf{n}\times\varphi_\tau],\nu)\,dl$$

$$= -\lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_C(\varphi_\tau, l)\,dl = -([\nabla^S\times\varphi_\tau], n)\equiv 0 \quad . \tag{17}$$

Similarly,

$$\left(\nabla^S, \left[\mathbf{n}\times\frac{\varphi_\tau}{|\mathbf{r}-\mathbf{r}'|}\right]\right) = \left(-\left[\nabla^S\times\frac{\varphi_\tau}{|\mathbf{r}-\mathbf{r}'|}\right], \mathbf{n}\right) =$$

$$= -\left(\left[\nabla\times\frac{\varphi_\tau}{|\mathbf{r}-\mathbf{r}'|}\right], \mathbf{n}\right). \tag{18}$$

Substituting (17) and (18) into (16) and then into (15), we get :

$$[\nabla'\times\mathbf{K}^S] = \frac{1}{4\pi}\nabla'\iint_S\left(\left[\nabla\times\frac{\varphi_\tau}{|\mathbf{r}-\mathbf{r}'|}\right], n\right)dS \equiv 0 \tag{19}$$

$$\mathbf{r}'\notin S$$

(because the surface integral on the right-hand side of (19), due to the Stokes theorem, reduces to a curvilinear integral over the boundary of S which is a closed surface).

Thus, everywhere outside S the function $\mathbf{K}^S$ describes a Laplace vector field :

$$(\nabla, \mathbf{K}^S)\equiv 0 \quad ; \quad [\nabla\times\mathbf{K}^S]\equiv 0 \quad ; \quad \mathbf{r}'\notin S\,, \tag{20}$$

and the scalar components of $\mathbf{K}^S$ are harmonic functions (in general, different on different sides of S). This property of (12) and the fact that in a two-dimensional case the right-hand side of (12) reduces to a complex Cauchy integral gives us ground to call (12) the *three-dimensional Cauchy integral analog,* and the function φ, its *vector density.*

2. Properties of Three-dimensional Cauchy Integral Analogs

Similarity of integrals (12) and (2) is manifested in the commonness of their properties. Indeed, simple calculations show that the Cauchy integral analogs have the same properties as the classical integrals. The following are the most important properties.

1st property follows from (20), i.e everywhere outside S the function $\mathbf{K}^S(\mathbf{r}', \varphi)$ can be expressed as the gradient of some harmonic functions $\Psi^\pm(\mathbf{r}')$ (different on different sides of S) :

$$\mathbf{K}^S(\mathbf{r}', \varphi) = \nabla\Psi^\pm(\mathbf{r}') \tag{21}$$

(where the "+" sign relates to the domain D, while "–" to $C\bar{D}$). By means of simple transformations, we can derive an explicit expression for Ψ. Indeed, by virtue of (11)

$$\varphi_\tau = \nabla^S\varphi \tag{22}$$

where φ is some function twice continuously differentiable on S, which we shall call the *scalar density* of the Cauchy integral. Using the Stokes theorem, and since S is a closed surface, we can transform the second integral in (13) as follows :

$$-\frac{1}{4\pi}\left[\nabla'\times\iint_S [\mathbf{n}\times\varphi_\tau]\frac{1}{|\mathbf{r}-\mathbf{r}'|}dS\right] =$$

$$= \frac{1}{4\pi}\left[\nabla'\times\iint_S\frac{[\nabla^S\varphi\times\mathbf{n}]}{|\mathbf{r}-\mathbf{r}'|}dS\right] \tag{23}$$

$$= \frac{1}{4\pi}\nabla'\iint_S\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \varphi\mathbf{n}\right)dS$$

Let us introduce the following notations :

$$q^S = -(\mathbf{n}, \varphi_{\mathbf{n}}) \tag{24}$$

$$\mathbf{m}^S = -\varphi\cdot\mathbf{n}$$

Now, substituting (23) into (13), by virtue of (24), we obtain :

$$\mathbf{K}^S(\mathbf{r}', \varphi) = \nabla'\left\{-\frac{1}{4\pi}\iint_S\frac{q^S}{|\mathbf{r}-\mathbf{r}'|}dS - \frac{1}{4\pi}\iint_S\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}, m^S\right)dS\right\} \tag{25}$$

Hence

$$\Psi^\pm(\mathbf{r}') = -\frac{1}{4\pi}\iint_S\frac{q^S}{|\mathbf{r}-\mathbf{r}'|}dS - \frac{1}{4\pi}\iint_S\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}, m^S\right)dS \tag{26}$$

Thus, the three-dimensional Cauchy integral analog can be expressed as the sum of a field of a simple layer of surface density $q^S = (\mathbf{n}, \varphi_n)$ and a field of a double layer of dipole moment $m^S = -\varphi\cdot\mathbf{n}$. This representation is analogous to the representation of real and imaginary parts of the classical Cauchy integral in the form of superposition of simple and double layer fields (Gakhov 1963).

2nd property

If a point $\mathbf{r}'$ lies on the surface S, i.e. $\mathbf{r}' = \mathbf{r}^0 \in S$ the Cauchy integral is an improper integral because the integrand tends to infinity when $\mathbf{r} = \mathbf{r}^0$. It can be calculated as the limit :

$$\mathbf{K}^S(\mathbf{r}^0, \varphi) = \lim_{\rho \to 0} \mathbf{K}^{S_\rho}(\mathbf{r}^0, \varphi) \tag{27}$$

where S_ρ is a part of S outside a sphere O_ρ of radius ρ and center at the point $\mathbf{r}^0$. This limit is called the *singular integral* in the sense of Cauchy principle value.

It has been shown (Zhdanov, 1974) that if S is a Lyapunov smooth surface and if the vector density of a Cauchy integral satisfies the Hölder boundary condition on S, i.e. if there exist positive constants L and h $(0 < h \leqslant 1)$ such that

$$|\varphi(\mathbf{r}') - \varphi(\mathbf{r})| \leqslant L\,|\mathbf{r}' - \mathbf{r}|^h, \quad \mathbf{r}, \mathbf{r}' \in S \tag{28}$$

then the limit (27) exists and is equal to

$$\mathbf{K}^S(\mathbf{r}^0, \varphi) = \mathbf{K}^S(\mathbf{r}^0, \varphi - \varphi(\mathbf{r}^0)) + \frac{1}{2}\,\varphi(\mathbf{r}^0) \tag{29}$$

where $\mathbf{K}^S(\mathbf{r}^0, \varphi - \varphi(\mathbf{r}^0))$ is calculated as an ordinary integral (because the integrand does not tend to infinity at any point).

3rd property

The behaviour of three-dimensional Cauchy integral analogs near the integration surface is quite important in the theory of potentials. We can show that in those cases where the density φ satisfies the Hölder boundary condition, there do exist limits for the Cauchy integral analog on approaching the surface S from either side :

$$\mathbf{K}^+(\mathbf{r}^0, \varphi) = \lim_{\substack{\mathbf{r}' \to \mathbf{r}^0 \\ \mathbf{r}' \in D}} \mathbf{K}^S(\mathbf{r}', \varphi) = \mathbf{K}^S(\mathbf{r}^0, \varphi) + \frac{1}{2}\,\varphi(\mathbf{r}^0)\,;$$
$$\mathbf{K}^-(\mathbf{r}^0, \varphi) = \lim_{\substack{\mathbf{r}' \to \mathbf{r}^0 \\ \mathbf{r}' \notin \bar{D}}} \mathbf{K}^S(\mathbf{r}', \varphi) = \mathbf{K}^S(\mathbf{r}^0, \varphi) - \frac{1}{2}\,\varphi(\mathbf{r}^0) \tag{30}$$

These limits are, however, different on different sides ; therefore a jump in value takes place on crossing the surface. The jump is equal to the density in magnitude :

$$\mathbf{K}^+(\mathbf{r}^0, \varphi) - \mathbf{K}^-(\mathbf{r}^0, \varphi) = \varphi(\mathbf{r}^0) \tag{31}$$

Eqs. (30) and (31) are the analogs of the Sokhotskiy-Plemel formula (*).

4th property

If φ is the limit on S of the gradient of a function everywhere harmonic in D, then by the Cauchy integral formula :

$$\mathbf{K}^S(\mathbf{r}'\varphi) = \begin{cases} \varphi(\mathbf{r}')\,; & \mathbf{r}' \in D \\ 0\,; & \mathbf{r}' \in C\bar{D} \end{cases} \tag{32}$$

(*) The second and the third properties may be considered to be the corollaries of (21) and (26) and the well known properties of simple and double layer potentials (Tikhonov, Samarskiy, 1953).

5th property

In order that a function $\varphi(\mathbf{r})$ be the limit on S of the gradient of a certain function harmonic in D, it is necessary and sufficient that

$$\mathbf{K}^S(\mathbf{r}', \varphi) \equiv 0\,; \quad \mathbf{r}' \notin \bar{D} \tag{33}$$

These properties of the Cauchy integral analogs are similar to the properties of the usual Cauchy type integral. Moreover, it is not difficult to show, as we have done in Appendix A for the Cauchy integral theorem, that in two-dimensional cases (where $\varphi = (\varphi_x, 0, \varphi_r)$ is not dependent on y) the integral (12) simply reduces to the classical Cauchy type integral (2), when $\varphi(\zeta) = -\varphi_x(x, \mathrm{r}) + i\varphi_{\mathrm{r}}(x, \mathrm{r})$.

3. Analytical Continuation of Three-dimensional Cauchy Integral Analogs through Integration Surface

The integration surface is, as already mentioned, a special surface for the functions described by Cauchy type integrals. In many cases, however, the Cauchy type integral can be analytically continued beyond the integration surface. This problem has important significance in geophysical applications. Consider a certain analytical part Γ of a surface S on which the Cauchy integral analog (12) of vector density $\varphi(\mathbf{r})$ is defined. We shall assume that the components of $\varphi(\mathbf{r})$ on Γ are described by analytical functions of coordinates and that they satisfy the Hölder boundary condition (28). Take a certain fixed point $\mathbf{r}^0 \in \Gamma$. Then, according to the Cauchy theorem (Smirnov, Sretenskiy, 1946), in the neighbourhood of $\mathbf{r}^0$ there exists a harmonic function $\Phi(\mathbf{r})$ satisfying the condition :

$$\left.\frac{\partial \Phi}{\partial \mathbf{n}}\right|_\Gamma = \varphi_{\mathbf{n}}\,; \quad \Phi|_\Gamma = \varphi\,, \tag{34}$$

on Γ, where φ is the scalar density of the Cauchy type integral which is related to the vector density by Eq. (22) ; hence

$$\nabla \Phi|_\Gamma = \boldsymbol{\varphi} \tag{35}$$

Now we shall use the Sokhotskiy – Plemel formula for the Cauchy integral analogs. By virtue of (31) and (35), we have

$$\mathbf{K}^-(\mathbf{r}^0, \varphi) = \mathbf{K}^+(\mathbf{r}^0, \varphi) - \nabla\Phi(\mathbf{r}^0) \quad \mathbf{r}^0 \in \Gamma \tag{36}$$

Obviously, the right-hand side of (36) is the boundary value on Γ of the gradients of functions harmonic in some domain D^+ adjacent to Γ and wholly lying inside D. The left-hand side describes the boundary value of the gradients of fields everywhere harmonic outside D. Consequently, the right-hand side is, according to the Stal theorem (Sretenskiy, 1946), the analytical continuation of the left-hand side through the surface Γ.

Analytical continuation of Cauchy type integrals from inside to outside of an analytical part of the surface Γ is demonstrated in a similar manner.

From (36) it is seen that $\mathbf{K}^{\text{cont}}(\mathbf{r}, \varphi)$, which are the values of the Cauchy integral analogs continued across the surface, differ from the integrals by a quantity equal to $-\nabla\Phi(\mathbf{r})$:

$$\mathbf{K}^{\text{cont}}(\mathbf{r}, \varphi) - \mathbf{K}(\mathbf{r}, \varphi) = -\nabla\Phi(\mathbf{r}) \qquad (37)$$

Hence, we obtain two corollaries : first, the singular points of $\mathbf{K}^{\text{cont}}$ are the same as the singular points of $\nabla\Phi(\mathbf{r})$. Second, the joining lines of different analytical parts of the surface S are the branching lines of the Cauchy integral analogs. These properties of the Cauchy integral analogs give us a means to study the analytical continuation of external geopotential fields inside perturbing masses.

4. Representation of Gravitational Fields in the form of Three-dimensional Cauchy Integral Analogs

Remarkable advances made in theoretical analysis of plane problems of gravimetry and magnetometry have been achieved, as already mentioned elsewhere, due to the representation of gravitational and magnetic fields in the form of Cauchy type integrals. These representations were first derived by A.V. Tsirulskiy (1963) and were then further developed and generalized by G. Ya. Golizdra (1966), and V.N. Strakhov ($1970_{1,2,3}$). In this connection, it is quite tempting to derive similar representations for three-dimensional fields. These problems were first solved by Zhdanov (1973, 1974) who, using matrix formalism, gave the representation for gravitational fields of uniformly perturbed bodies in the form of Cauchy integral analogs. Later T. Kolbenheyer (1976, 1978) studied the description of gravitational field of a body with an arbitrary density distribution $\sigma(\mathbf{r})$, using the Moisil-Teodoresko theory (Bitsadze, 1953, 1972). In the present paper we shall solve this problem with the help of vector representation of Cauchy type integral developed in the previous pages.

We shall begin our analysis with gravitational fields. The intensity of gravitational fields of a domain D bounded by a piecewise smooth surface S and filled with masses distributed in D with an arbitrarily continuous density $\sigma(\mathbf{r})$, as is known, has the form :

$$\mathbf{U}(\mathbf{r}') = \gamma \iiint_D \sigma(\mathbf{r}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV_{\mathbf{r}} \qquad (38)$$

where γ is the gravitational constant. We shall now reduce the volume integral in (38) to a Cauchy type integral defined on the surface S bounding the domain D. For this purpose, using the Kolbenheyer technique (1978), we shall extend the definition of the function $\sigma(\mathbf{r})$ into some domain D^* wholly containing the surface S and the domain D ($D \subset D^*$; $S \subset D^*$), such that $\sigma(\mathbf{r})$ is continuous in D^*.

Let $h(\mathbf{r})$, $\mathbf{r} \in D^*$ be an arbitrary particular solution of the Poisson equation :

$$\Delta h(\mathbf{r}) = -4\pi\gamma\sigma(\mathbf{r}) \qquad (39)$$

and

$$f(\mathbf{r}) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \qquad (40)$$

the Green function. Substituting (39) and (40) into (7), we obtain the representation for the external gravitational field of the domain D in the form of Cauchy integral analog :

$$\mathbf{U}(\mathbf{r}') = -\gamma \iiint_D \sigma(\mathbf{r}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV_{\mathbf{r}} = \mathbf{K}^S(\mathbf{r}', -\nabla h), \qquad (41)$$

where $\mathbf{r}' \in C\bar{D}$.

Now we shall introduce analogous representations for the inner gravitational field $\mathbf{W}(\mathbf{r}')$ of the domain D. Obviously,

$$\mathbf{W}(\mathbf{r}') = \nabla' \varphi(\mathbf{r}') \qquad (42)$$

where

$$\Delta' \varphi(\mathbf{r}') = -4\pi\gamma\sigma(\mathbf{r}') \quad \mathbf{r}' \in D \qquad (43)$$

By virtue of (39), the difference $\varphi(\mathbf{r}') - h(\mathbf{r}')$ is a function harmonic in D :

$$\Delta(\varphi - h) = 0 \qquad \mathbf{r}' \in D$$

Hence, by the fourth property of the Cauchy integral analog, we have

$$\begin{aligned}\mathbf{W}(\mathbf{r}') - \nabla' h(\mathbf{r}') &= \mathbf{K}^S(\mathbf{r}', \mathbf{W}(\mathbf{r}) - \nabla h(\mathbf{r})) = \\ &= \mathbf{K}^S(\mathbf{r}', \mathbf{W}(\mathbf{r})) + \mathbf{K}^S(\mathbf{r}', -\nabla h(\mathbf{r})) ; \qquad (44)\\ &\text{for } \mathbf{r}' \in D\end{aligned}$$

We know that the intensity of a gravitational field due to continuously distributed masses is a everywhere continuous function. Consequently,

$$\mathbf{U}(\mathbf{r}) \equiv \mathbf{W}(\mathbf{r}) \qquad \text{for } \mathbf{r} \in S$$

and

$$\mathbf{K}^S(\mathbf{r}', \mathbf{W}(\mathbf{r})) = \mathbf{K}^S(\mathbf{r}', \mathbf{U}(\mathbf{r})) = 0 ; \qquad (45)$$

$$\text{for } \mathbf{r}' \in D$$

because $\mathbf{U}(\mathbf{r})$ is harmonic outside D. Substituting (45) into (44), we can rewrite

$$\mathbf{W}(\mathbf{r}') = \nabla' h(\mathbf{r}') + \mathbf{K}^S(\mathbf{r}', -\nabla h(\mathbf{r})) ; \qquad (46)$$

$$\mathbf{r}' \in D$$

ow consider the function :

$$\chi(\mathbf{r}) = (\mathbf{r}\,,\nabla' h(\mathbf{r}')) \quad (47)$$

bviously,

$$\begin{matrix} \nabla\chi(\mathbf{r}) \equiv \nabla' h(\mathbf{r}) \\ \Delta\chi(\mathbf{r}) \equiv 0 \end{matrix} \quad \mathbf{r}\in D \quad (48)$$

lence, by the fourth property of the Cauchy type integrals, we have

$$\mathbf{K}^S(\mathbf{r}\,,\nabla\chi(\mathbf{r})) = \mathbf{K}^S(\mathbf{r}'\,,\nabla' h(\mathbf{r}')) = \begin{cases} 0\ ; & \mathbf{r}'\in C\bar{D}\ ; \\ \nabla' h(\mathbf{r}')\ ; & \mathbf{r}'\in D\ ; \end{cases} \quad (49)$$

By virtue of (49), Eqs. (41) and (46) can be combined into one expression :

$$\mathbf{K}^S(\mathbf{r}'\,,[\nabla' h(\mathbf{r}') - \nabla h(\mathbf{r})]) = \begin{cases} \mathbf{U}(\mathbf{r}')\ ; & \mathbf{r}'\in C\bar{D} \\ \mathbf{W}(\mathbf{r}')\ ; & \mathbf{r}'\in D \end{cases} \quad (50)$$

Thus, we have derived expressions for gravitational fields both inside and outside the perturbing masses in terms of the same Cauchy type integral (50). These expressions extend the Strakhov formula (1970) to three-dimensional case.

Now we shall examine how these expression (41) and (50) can be simplified for an important particular case of bodies with a uniform or linear density. Let

$$\sigma(\mathbf{r}) \equiv \sigma_0 = \text{const}\,.$$

Then, we can take

$$h = -2\pi\gamma\frac{\sigma_0}{3}\mathbf{r}^2 \quad (51)$$

as the particular solution of Eq. (39). Accordingly, Eq. (41) can be rewritten as

$$\mathbf{U}(\mathbf{r}) = \mathbf{K}^S\left(\mathbf{r}'\,,\frac{4\pi}{3}\gamma\sigma_0\,\mathbf{r}\right) \quad (52)$$

This expression extends the Tsirulskiy formula for a plane field (Tsirulskiy, 1963), to three-dimensional case :

$$U(\zeta') = \frac{1}{2\pi i}\int_C \frac{-2\pi\gamma\sigma_0\,\zeta^*}{\zeta - \zeta'}\,d\zeta \quad (53)$$

where ζ^* is the conjugate of ζ, C is the cross-section of a two-dimensional body extended along Y-axis, $U(\zeta')$ is the complex intensity of the gravitation field related to $\mathbf{U}(U_x\,,O\,,U_z)$ by means of the expression :

$$U(\zeta') = -U_x(x'\,,r') + i\,U_z(x'\,,r') \quad (54)$$

By virtue of the results stated in Append. A, the expression (52) in two-dimensional cases is readily converted into (53) when 2π is substituted for $4\pi/3$ (because a finite three-dimensional domain is converted into a two-dimensional region, i.e. infiinitely extended along Y-axis).

Expression (50) for a uniform body is written as :

$$K^S\left(\mathbf{r}'\,,\frac{4\pi}{3}\gamma\sigma_0(\mathbf{r}-\mathbf{r}')\right) = \begin{cases} \mathbf{U}(\mathbf{r}')\ ; & \mathbf{r}'\in C\bar{D} \\ \mathbf{W}(\mathbf{r}')\ ; & \mathbf{r}'\in D \end{cases} \quad (55)$$

On expanding the left-hand side, with the help of (12), we can finally obtain :

$$\frac{\gamma\sigma_0}{3}\iint_S \left\{ \frac{2(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{n}\,,\mathbf{r}-\mathbf{r}') - \frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|} \right\} dS = \begin{cases} \mathbf{U}(\mathbf{r}')\ ; & \mathbf{r}'\in C\bar{D} \\ \mathbf{W}(\mathbf{r}')\ ; & \mathbf{r}'\in D \end{cases} \quad (56)$$

Now consider a case where the density $\sigma(\mathbf{r})$ varies according to a linear law :

$$\sigma(\mathbf{r}) = (C\,,\mathbf{r}) + \sigma_0 \quad (57)$$

where : $4\pi\gamma\mathbf{C} = -(C_x\,,C_y\,,C_z)$, σ_0 are certain arbitrary constant vector and scalar, respectively. Here we have to take

$$\mathrm{h}(\mathbf{r}) = \frac{1}{6}(C_x\,.\,x^3 + C_y\,y^3 + C_z\,.\,z^3) - \frac{2\pi\gamma\sigma_0}{3}r^2\,, \quad (58)$$

as the particular solution of (39). Consequently, Eq. (41) has to be rewritten as

$$\mathbf{U}(\mathbf{r}') = \mathbf{K}^S\left(z'\,.\left\{-\frac{1}{2}[x^2C_x\mathbf{e}_x + y^2C_y\mathbf{e}_y + z^2C_z\mathbf{e}_z] + \frac{4\pi}{3}\gamma\sigma_0\,\mathbf{r}\right\}\right) \quad (59)$$

while (50) takes the form :

$$\mathbf{K}^S\left(\mathbf{r}'\,,\left\{\frac{4\pi}{3}\gamma\sigma_0(\mathbf{r}-\mathbf{r}') - \frac{1}{2}[(x^2 - x'^2)\,C_x\,\mathbf{e}_x + (y^2 - y'^2)\,.\,C_y\mathbf{e}_y + (z^2 - z'^2)\,C_z\mathbf{e}_z]\right\}\right) = \begin{cases} \mathbf{U}(\mathbf{r}')\ ; & \mathbf{r}'\in C\bar{D} \\ \mathbf{W}(\mathbf{r}')\ ; & \mathbf{r}\in D \end{cases} \quad (60)$$

This expression is the three-dimensional analog of Strakhov's formula (Strakhov, 1970). Evidently, using similar arguments, we can easily construct the representation for gravitational fields in the form of Cauchy integral analogs for bodies whose density changes according to any polynomial low.

Such a representation may be quite useful in solving the direct problems of gravimetry, because by means of these representations we can easily convert the volume integrals into surface integrals even for nonuniform bodies. They have, however, the greatest value in theoretical investigations or in studying the analytical continuation of gravitational fields inside or outside masses.

5. Representation of a Magnetic Field in the Form of Cauchy Integral Analog

The intensity of magnetic field of a domain D, in which the magnetic masses of arbitrarily continuous magnetization $\mathbf{J}(\mathbf{r})$ are distributed, is of the form :

$$\mathbf{H}(\mathbf{r}) = \nabla' \iiint_D \left(\mathbf{J}(\mathbf{z}) , \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV_r \quad (61)$$

Proceeding as in the previous Section, we can reduce the volume integral in (61) to a Cauchy integral analog over a surface S bouding the domain D. For this purpose transform (61) by the usual technique :

$$\mathbf{H}(\mathbf{r}') = \nabla' \left\{ \iiint_D \frac{(\nabla , \mathbf{J})}{|\mathbf{r} - \mathbf{r}'|} dV_r - \iint_S \frac{(\mathbf{n} , \mathbf{J})}{|\mathbf{r} - \mathbf{r}'|} dS_r \right\} =$$

$$= - \iiint_D (\nabla , \mathbf{J}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV_r +$$

$$+ \iint_S (\mathbf{n} , \mathbf{J}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS_r \quad (62)$$

The magnetization $\mathbf{J}$ being potential in D, we can take :

$$\mathbf{J} = - \nabla h \quad (63)$$

Hence, with the help of (7) we can rewrite the first integral in (62) as follows :

$$- \iiint_D (\nabla , \mathbf{J}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV_r =$$

$$= - \iint_S (\mathbf{n} , \mathbf{J}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS_r -$$

$$- \iint_S \left[[\mathbf{n} \times \mathbf{J}] \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] dS_r \quad (64)$$

Substituting (64) into (62), we obtain :

$$\mathbf{H}(\mathbf{r}') = - \iint_S \left[[\mathbf{n} \times \mathbf{J}] \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] dS_r =$$

$$= \mathbf{K}^S (\mathbf{r}' , 4\pi \mathbf{J}_\tau) , \quad (65)$$

where $\mathbf{J}_\tau$ is the component of $\mathbf{J}$ tangential to S.

The expression (65) shows that for a wide class of magnetization distributions inside D, the magnetic field of D can be expressed in the form of a Cauchy integral analog over the surface of S.

In a particular case, where the function h in (63) is harmonic in D, by virtue of (62) and (65), we have :

$$\mathbf{H}(\mathbf{r}') = \iint_S (\mathbf{n} , \mathbf{J}) \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS = \mathbf{K}^S (\mathbf{r}' , 4\pi \mathbf{J}\mathbf{n}) \equiv$$

$$\equiv \mathbf{K}^S (\mathbf{r}' , 4\pi \mathbf{J}_\tau) \quad (66)$$

where J_n is the component of J normal to the surface S. The expressions (65) and (66), just like (41), give a basis with which we can solve the problem of analytical continuation of external field inside a magnetized body.

6. Analytical Continuation of An External Gravitational Or Magnetic Field inside a Three-diemensional Domain containing the Field Sources

The representations derived in the previous Section for external gravitational and magnetic fields in the form of Cauchy integral analogs lie at the base of the theory of analytical continuation of fields inside and outside masses.

Let Γ be some analytical part of the surface S and let the density $\sigma(\mathbf{r})$ and magnetization $\mathbf{J}(\mathbf{r})$ be analytical functions of coordinates everywhere inside $\bar{D}$ (domain D along with its boundary S). The functions $\nabla h|_\Gamma$ and $\mathbf{J}_\tau|_\Gamma$ are evidently analytical on Γ. Hence, by the Cauchy theorem, in the neighbourhood of Γ there exist such harmonic functions $\Phi^U(\mathbf{r})$ and $\Phi^H(\mathbf{r})$ that on Γ :

$$\nabla \Phi^U (\mathbf{r})|_\Gamma = - \nabla h|_\Gamma \quad (67a)$$

$$\nabla \Phi^H (\mathbf{r})|_\Gamma = 4\pi \mathbf{J}_\tau|_\Gamma \quad (67b)$$

For example, if the domain D is filled with masses of uniform density σ_0, Eq. (67a), by virtue of (51), can be rewritten as :

$$\nabla \Phi^U (\mathbf{r})|_\Gamma = \frac{4\pi}{3} \gamma \sigma_0 \mathbf{r} \quad (68)$$

It is called the equation of the surface Γ in a harmonic form (Zhdanov, 1974). In a two-dimensional ase, where S is a cylindrical surface with its generator arallel to Y-axis, Eq. (68) is converted into the equa- on of a plane curve in the form of Tsirulskiy (1963) :

$$\zeta^* = \Psi(\zeta) = \frac{3}{4\pi\gamma\sigma_0}\left[\frac{\partial\Phi^U}{\partial x}(x, z) - i\frac{\partial\Phi^U}{\partial \mathbf{r}}(x, z)\right] \quad (69)$$

/here ζ^* is the conjugate of $\zeta = x + iz$, $\Psi(\zeta)$ is a omplex analytical function.

Owing to Eq. (67) which defines the equation of he analytical part Γ of the surface S, we can make se of the properties of the Cauchy integral analog i ∇h and $\mathbf{J}_\tau$ satisfy the H¨older boundary condi- ion on Γ), according to which external gravitational nd magnetic fields (described by the Cauchy integral nalogs) can be extended within the masses. Moreover, y virtue of (37), (41) and (65), the continued field alues are given by the formulas :

$$\mathbf{U}(\mathbf{r}) = -\nabla\Phi^U(\mathbf{r}) + \mathbf{F}^U(\mathbf{r}) \quad (70a)$$

$$\mathbf{H}(\mathbf{r}) = -\nabla\Phi^H(\mathbf{r}) + \mathbf{F}^H(\mathbf{r}) \quad (70b)$$

/here

$$\mathbf{F}^U(\mathbf{r}) = \mathbf{K}^S(\mathbf{r}, -\nabla h) = \mathbf{K}^S(\mathbf{r}, \nabla\Phi^U);$$
$$\mathbf{F}^H(\mathbf{r}) = \mathbf{K}^S(\mathbf{r}, 4\pi\mathbf{J}_\tau) = \mathbf{K}^S(\mathbf{r}, \nabla\Phi^H) \quad (71)$$

'his expression gives a comprehensive means to study he analytical continuation of external field into the omain D. In particular, if the whole surface S is ana- ytical, and the distributions of $\sigma(\mathbf{r})$, $\mathbf{J}(\mathbf{r})$ are described y analytical functions, then the following theorems, /hich extend the theorem of Strakhov (1970)(*) to h ee-dimensional case, hold true.

'heorem 1 : If S is a closed analytical surface given y Eq. (67), the functions $\Phi^U(\mathbf{r})$ and $\Phi^H(\mathbf{r})$ have ne- essarily singular points inside S, and the number of heir singularities is equal to the number of singula- ities of the function $\mathbf{U}(\mathbf{r})$ or $\mathbf{H}(\mathbf{r})$ which are the ana- ytical continuations into S.

'heorem 2 : In order that two different domains D_1 r.d D_2 field with masses of different analytical dis- r:butions of densities $\sigma_1(\mathbf{r})$ and $\sigma_2(\mathbf{r})$ or magnetizations $_1(\mathbf{r})$ and $\mathbf{J}_2(\mathbf{r})$, and bounded by analytical surfaces S_1 nd S_2, may create identically equal external fields :

$$\mathbf{r} \in CD_0 \qquad \mathbf{U}^{(1)}(\mathbf{r}) \equiv \mathbf{U}^{(2)}(\mathbf{r}); \quad (72a)$$
$$\mathbf{H}^{(1)}(\mathbf{r}) \equiv \mathbf{H}^{(2)}(\mathbf{r}); \quad (72b)$$

t is necessary and sufficient that the domains D_1 and). intersect and the functions $\Phi^{U(1)}$ and $\Phi^{U(2)}$ or $\Phi^{H(1)}$ and $\Phi^{H(2)}$ defining the surfaces S_1 and S_2 have singularities only in the intersection $D_0 = D_1 \cap D_2$; moreover, the differences $\Phi^{U(1)} - \Phi^{U(2)} = \delta\Phi^U$ or $\Phi^{H(1)} - \Phi^{H(2)} = \delta\Phi^H$ be functions harmonic in D_0. The truth of Theorem 1 follows directly from Eqs. (70) and (71). We shall prove Theorem 2 in Appendix B.

In the general case where the boundary S of the domain is a piecewise analytical surface, the following statements which are the generalizations of Strakhov theorem (1970) for plane fields to three-dimensional case are true :

Theorem 3 : An external gravitational or magnetic field generated by analytical distributions of density or ma- gnetization inside a domain D (such that $\mathbf{J} = -\nabla h$ in D and the functions $\nabla h\,|_S$ and $\mathbf{J}_\tau\,|_S$ satisfy the Holder boundary condition on the boundary S of D) admits analytical continuation through any analytical part Γ of the surface S, besides, perhaps, its boundary. The singular points of the continued field are the same as the singular points of functions describing the equation of the surfaces Γ in the form (67).

Theorem 4 : If the boundary S of a domain D consists of a finite number of different analytical parts Γ_i ($i = 1,2,3, \ldots, N$) ("different" in the sense that dif- ferent analytical functions $\Phi^{(i)}(\mathbf{r})$, ($i = 1,2,3, \ldots, N$) occur in the equations of these surfaces (67)), then for any analytical distribution of the density or magne- tization inside D (such that $\mathbf{J} = -\nabla h$ in D and $\nabla h\,|_S$ and $\mathbf{J}_\tau\,|_S$ satisfy the Hölder boundary condition on S) the joining lines of different analytical parts Γ_i and Γ_j are the branching lines of the external field.

The proofs of Theorems 3 and 4 follow directly from (70) and (71). These expressions (70) and (71) can also be used in solving certain inverse problems of gravimetry and magnetometry, employing the prin- ciples of analytical continuation. Such problems were first studied by A.A. Zamorev (1941 ; 1942) and A.V. Tsirulskiy (1964) for uniform two-dimensional bodies, and were later extended by Strakhov (1970_2) to two- dimensional bodies of arbitrary analytical magnetization distribution. Zhdanov (1974) applied a similar technique to solve the three-dimensional inverse problems in the theory of potentials for bodies of uniform density. It should, however, be noted that the use of these methods for solving the inverse problems is rather cumbersome due to the instability of the analytical continuation procedure. They have their main significance in theo- retical investigations because they reveal the conditions under which the solution of the inverse problem is unique. We shall extend these results to a three-dimen- sional body of arbitrary analytical density distribution.

Let a surface S be composed of several analytical parts Γ_i ($i = 1,2,3, \ldots, N$) and assume that we know the analytical density distribution inside S (which is

(*) For a domain of uniform density or magnetization hese theorems were first derived and demonstrated for a two- imensional case by A.V. Tsirulskiy (1963 ; 1969), and extend- d to three-dimensional case by Zhdanov (1973).

such that the gradient of the corresponding function $h(\mathbf{r})$, a particular solution of Eq. (39), satisfies the Hölder boundary condition on S). Besides, we shall assume that the analytical function $\Phi^{U(1)}$ defining the equation of the surface Γ_i in the form (67a) is known. It is assumed that this function is analytical everywhere inside D (except, perhaps, at certain finite number of isolated singular points). The problem is to find the whole surface S from the external gravitational field $\mathbf{U}(\mathbf{r})$, using the analytical continuation technique.

The solution of this problem follows from (70a). Indeed, let $\mathbf{U}^{(i)}(\mathbf{r})$ denote the analytical continuation of the external gravitational field across the surface Γ_i. Then

$$\mathbf{F}^U(\mathbf{r}) = \mathbf{U}^{(i)}(\mathbf{r}) + \nabla\Phi^{U(i)}(\mathbf{r}) \,; \quad i = 1,2,\ldots,N \qquad (73)$$

In particular, for a known part Γ of the surface S:

$$\mathbf{F}^U(\mathbf{r}) = \mathbf{U}^{(1)}(\mathbf{r}) + \nabla\Phi^{U(1)}(\mathbf{r}) \qquad (74)$$

Equating the right-hand sides of (73) and (74), we get

$$\nabla\Phi^{U(i)}(\mathbf{r}) = \nabla\Phi^{U(1)}(\mathbf{r}) - \mathbf{U}^{(i)}(\mathbf{r}) + \mathbf{U}^{(1)}(\mathbf{r}) \qquad (75)$$

Consequently, the equation of the surface Γ_i, by virtue of (67), takes the form :

$$\nabla\Phi^{U(1)}(\mathbf{r}) - \mathbf{U}^{(i)}(\mathbf{r}) + \mathbf{U}^{(1)}(\mathbf{r}) = -\nabla h(\mathbf{r}) \qquad (76)$$

where $h(\mathbf{r})$ is the particular solution of Eq. (39).

Expression (76) gives the solution to the problem of determination of the shape of a surface composed of a finite number of analytical parts, on the basis of analytical continuation technique. This problem has evidently a unique solution, provided we not only know the density distribution inside S, but also some other part (however small it be) of this surface. In the contrary case, the inverse problem, as its follows from Theorem 2, is ambiguous.

Conclusion

The exemples given above do not form a complete list of the results which can be derived for three-dimensional fields with the help of Cauchy integral analog technique. Moreover, the inter-relations mentioned above between the Cauchy integral analogs and the classical theory of functions of a complex variable suggest that almost all the results obtained for the two-dimensional case can be extended to three-dimensional case as well. Thus, the sharp demarcation existing since long between the theoretical approaches applied for the interpretation of two-dimensional and three-dimensional geopotential fields has been obliterated. This fundamental result, that follows from the whole system of analogs constructed above, is quite important.

Appendix A

Illustration of the analogy between the Cauchy integral formula and Eq. (9) given in sec. 1

Introduce a righthanded Cartesian coordinate syste[m] xyz, the Z-axis being directed vertically downwards. L[et] S be a cylindrical surface generated by a line parallel t[o] Y-axis, and let the function h be not dependent on [y]. Then $\mathbf{n}\ (\mathbf{n}_x, 0, \mathbf{n}_z)$, $\nabla h = (h_x, 0, h_z)$ and the lef[t] hand side of (9) can be rewriten as :

$$-\frac{1}{4\pi}\iint_S \left\{ (\mathbf{n}\,,\nabla h)\,\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \left[[\mathbf{n}\times\nabla h]\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right] \right\} dS$$

$$= -\frac{1}{2\pi}\int_L \frac{(x'-x)(h_z dx - h_x dz) + (z'-z)(h_x dx + h_z dz)}{(x-x')^2 + (z-z')^2}\cdot \mathbf{e}_x$$

$$-\frac{1}{2\pi}\int_L \frac{(z'-z)(h_z dx - h_x dz) - (x'-x)(h_x dx + h_z dz)}{(x-x')^2 + (z-z')^2}\cdot \mathbf{e}_z\,,$$

where $\mathbf{e}_x$, $\mathbf{e}_z$ are the unit vectors along x and z, respe[c]tively ; $\mathbf{r}$ and $\mathbf{r}'$ have the coordinates $(x\,,z)$ and $(x', z[')]$ respectively ; and the contour L is the intersection [of] the surface S by the plane xz.

In the complex plane xz introduce the variab[le] $\zeta = x + iz$ and let $f(\zeta)$ stand for the complex functio[n]

$$f(\zeta) = -h_x(x\,,z) + ih_z(x\,,z) \qquad (\mathrm{A.}$$

Obviously, $f(\zeta)$ is analytical in D and continuous in [...] because h_x and h_z are interrelated by the Cauch[y-] Riemann condition due to the harmonicity of h.

Transforming (A.1) with the help of (A.2) and (9 we obtain the classical Cauchy integral formula :

$$\frac{1}{2\pi i}\int_L \frac{(-h_x + ih_z)}{(x-x') + i(z-z')}\, d(x+iz) = \qquad (\mathrm{A.}$$

$$= \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - \zeta'}\, d\zeta = \begin{cases} f(\zeta')\,; & \zeta' \in D\,; \\ 0\,; & \zeta' \in C\bar{D} \end{cases}$$

Thus, Eq. (9) is the natural generalization of the Cauc[hy] formula for the three-dimensional case ; it can, ther[e]fore, be called the analog of Cauchy formula.

Appendix B

Proof of the Theorem 2

(for the sake of brevity we shall consider the case of magnetized masses only)

Necessary condition : By virtue of Theorem 1, t[he] functions $\Phi^{H(1)}$ and $\Phi^{H(2)}$ should obviously have t[he] same singularities which should necessarily be insi[de] D_1 and D_2 ; consequently, either these domains shou[ld] intersect, or one should be contained wholly within t[he] other : $D_1 \cap D_2 = D_0 \neq 0$. For the sake of definiteness, we shall assume that $D \backslash D_0$ is never an empty set (i.e. either D_1 lies wholly within D_2, Fig. 1a, or D_1 intersects D_2, Fig. 1b). From (70b) and (71b), we have

$$\begin{aligned} \mathbf{H}^{(1)}(\mathbf{r}) &= -\nabla\Phi^{H(1)}(\mathbf{r}) + \mathbf{F}^{H(1)}(\mathbf{r}) ; \\ \mathbf{H}^{(2)}(\mathbf{r}) &= -\nabla\Phi^{H(2)}(\mathbf{r}) + \mathbf{F}^{H(2)}(\mathbf{r}) ; \end{aligned} \quad (B.1)$$

where the functions $\mathbf{F}^{H(1)}$ and $\mathbf{F}^{H(2)}$, being harmonic in D_1 and D_2, respectively, are given by the expressions :

$$\begin{aligned} \mathbf{F}^{H(1)}(\mathbf{r}) &= \mathbf{K}^{S_1}(\mathbf{r}, \nabla\Phi^{H(1)}) ; \\ \mathbf{F}^{H(2)}(\mathbf{r}) &= \mathbf{K}^{S_2}(\mathbf{r}, \nabla\Phi^{H(2)}) ; \end{aligned} \quad (B.2)$$

Obviously, the function $\mathbf{H}^{(1)}(\mathbf{r})$ is harmonic in $D_2 \backslash D_0$. Consequently, from the condition of the theorem, the function $\mathbf{H}^{(2)}(\mathbf{r})$ (or its analytical continuation) is also harmonic in $D_2 \backslash D_0$. Thus, by virtue of (B.1), $\nabla\Phi^{H(2)}$, being the difference of two harmonic functions, is itself harmonic in $D_2 \backslash D_0$. Therefore, using the fourth property of the Cauchy integral analog, we can write :

$$\mathbf{H}^{(2)}(\mathbf{r}') = \mathbf{K}^{S_2}(\mathbf{r}', 4\pi \mathbf{J}_\tau^{(2)}) = \mathbf{K}^{S_2}(\mathbf{r}', \nabla\Phi^{H(2)}) = = \mathbf{K}^{S_0}(\mathbf{r}', \nabla\Phi^{H(2)}), \quad (B.3)$$

where S_0 is the boundary of D_0, $\mathbf{r}' \in C\bar{D}_0$ (complement of the closed domain $\bar{D}_0 + S_0$ with respect to the whole space). Similarly, we can show that

$$\mathbf{H}^{(1)}(\mathbf{r}') = \mathbf{K}^{S_0}(\mathbf{r}', \nabla\Phi^{H(1)}) ; \quad \mathbf{r}' \in C\bar{D}_0 \quad (B.4)$$

Hence, we obtain

$$\mathbf{K}^{S_0}(\mathbf{r}', \nabla\Phi^{H(1)} - \nabla\Phi^{H(2)}) = \mathbf{K}^{S_0}(\mathbf{r}', \nabla(\delta\Phi^H)) \equiv 0 ; \quad \mathbf{r}' \in C\bar{D}_0 \quad (B.5)$$

Consequently, by the fifth property of Cauchy integral analogs, $\delta\Phi^H$ is a function harmonic everywhere inside D_0, which is what was required to be proved.

Sufficient condition : We shall now calculate the difference of the fields $\mathbf{H}^{(1)}(\mathbf{r}')$ and $\mathbf{H}^{(2)}(\mathbf{r}')$ by means of (B.3) and (B.4) :

$$\mathbf{H}^{(2)}(\mathbf{r}') - \mathbf{H}^{(1)}(\mathbf{r}') = \mathbf{K}^{S_0}(\mathbf{r}', \nabla(\delta\Phi^H)) ; \quad \mathbf{r}' \in C\bar{D}_0 \quad (B.6)$$

But, if $\delta\Phi^H$ is a function harmonic everywhere in D_0, by virtue of the fifth property of Cauchy integral analogs, the right-hand side of (B.6) identically vanishes. Consequently,

$$\mathbf{H}^{(1)}(\mathbf{r}') \equiv \mathbf{H}^{(2)}(\mathbf{r}') ; \quad \mathbf{r}' \in C\bar{D}_0 \quad (B.7)$$

Thus, Theorem 2 has been fully demonstrated.

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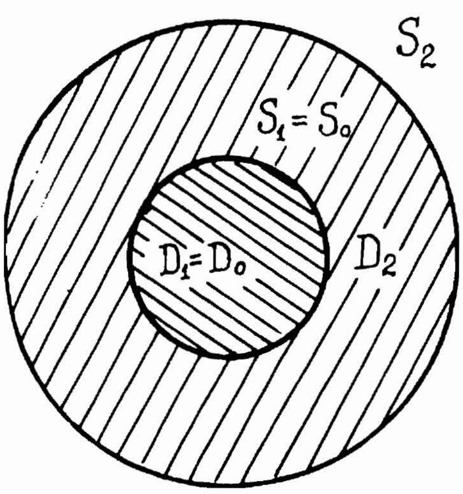

Fig. 1a

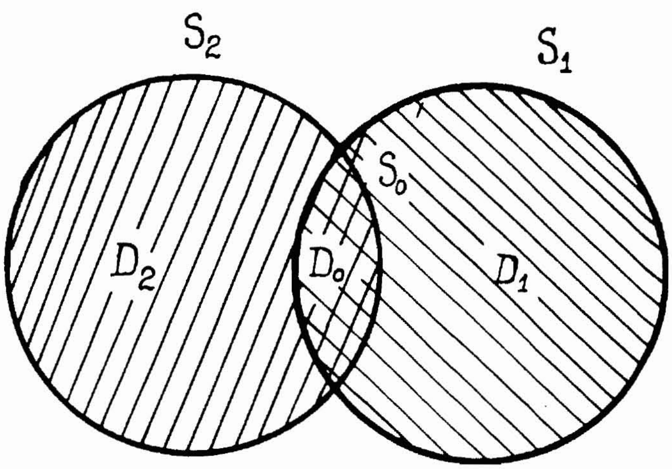

Fig. 1b

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