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The construction of effective methods for electromagnetic modelling

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Summary. This paper deals with the further development of finite-difference methods for electromagnetic field modelling in two- and three-dimensional cases. The main feature of the approach suggested here is the application of generalized asymptotic boundary conditions valid with the accuracy $\bar{O}(1/\rho^N)$, where ρ is the distance from the heterogeneities. The finite-difference approximation of problems under solution is made using the balance method, which results in 5-point difference schemes in the 2-D case and 7-point difference schemes in the 3-D case. To solve the linear system of difference equations the successive over-relaxation (SOR) method is used, the relaxation factor being chosen during the iteration procedure. In view of the vectorial character of the problem for the 3-D case, a successive blocked over-relaxation method (SBOR) is applied.

The model's validity is based on the comparison of the fields accounted at the ground surface with those computed by the integral transformation of excessive currents, determined in the heterogeneity region using the finite-difference scheme.

1 Introduction

One of the fundamental problems in modern geoelectrics is the construction of effective models for variable electromagnetic fields in inhomogeneous media. Different techniques have been applied to solve this problem: integral equations, finite differences, finite elements etc. These methods have advantages and disadvantages and, on the whole, the problem of choosing the optimal method for numerical modelling remains open. It seems important therefore to search for the most effective algorithms, to examine different approaches, and to select such classes of problems for which the application of these algorithms is most suitable. In this paper we will consider the principles of the construction of effective algorithms for finite-difference modelling of electromagnetic fields. This approach has been developed by many authors: Greenfield (1965); Swift (1971); Madden & Swift (1972); Jones & Pascoe (1971); Müller & Losecke (1975); Weaver & Brewitt-Taylor (1978); Tatrallyay (1977, 1978a); Judin & Kazantseva (1977); Dmitriev & Barashkov (1979); Barashkov (1980); Varentsov & Golubev (1980a, b, c); Lines & Jones (1973); Jones (1978); Judin (1980); Zhdanov & Spichak (1980); etc. A number of programs for finite-difference modelling are considered which are widely used in the practice of geoelectric research. In spite of the advances, however, some difficulties exist concerning the application of this technique, the resolution of which would significantly increase the effectiveness of such calculations. The main problems in our opinion, are as follow:

(1) the construction of accurate boundary conditions in the case when the distance between the boundary of the region modelled and the geoelectric heterogeneities is relatively small;
(2) the construction of correct finite-difference approximations of field equations;
(3) the selection of effective methods for solving the systems of finite-difference equations;
(4) the development of techniques for independent testing of model validity.

Note, that all the problems given above arise in solving both two-dimensional and three-dimensional modelling problems, however in the latter case they are considerably more complicated. On the other hand, the successful solution of these problems in three dimensions promises the greatest computational profits.

The first part of this paper considers in detail techniques for solving the 2-D problems mentioned above. In the second part we outline ways of generalizing these results for the 3-D case.

2 Two-dimensional modelling

2.1 THE GEOELECTRIC MODEL AND FIELD EQUATIONS

Consider the 2-D model, presented in Fig. 1, in which the conducting Earth involves the anomalous rectangular region V_a with arbitrary conductivity distribution $\sigma_a(x, z)$, surrounded by three regions of the normal section with a one-dimensional conductivity distribution: the region V_n^L with conductivity $\sigma_n^L(z)$ (the left normal section), the region V_n^R with conductivity $\sigma_n^R(z)$ (the right normal section) and the region V_n with conductivity $\sigma_n(z) \equiv \sigma_n^L(z) \equiv \sigma_n^R(z)$ (the lower normal section). The Earth is in contact with the non-conducting atmosphere $V_0\,(\sigma_0 = 0)$. The model is excited by the plane wave with E- or H-polarization. The time dependence of the field is specified as $\exp(-i\omega t)$, the displacement currents are ignored and the magnetic permeability is overall equal to μ_0 (the vacuum permeability).

In the case of E-polarization the modelled electric field satisfies the well-known equation:

$$\Delta E_y + K^2 E_y = 0 \tag{2.1}$$

where $K^2 = K^2(x, z) = i\omega\mu_0\sigma(x, z)$, and the associated magnetic field is determined by the electric field:

$$H_x = -\frac{1}{i\omega\mu_0}\frac{\partial E_y}{\partial z}$$

$$H_z = \frac{1}{i\omega\mu_0}\frac{\partial E_y}{\partial x}. \tag{2.2}$$

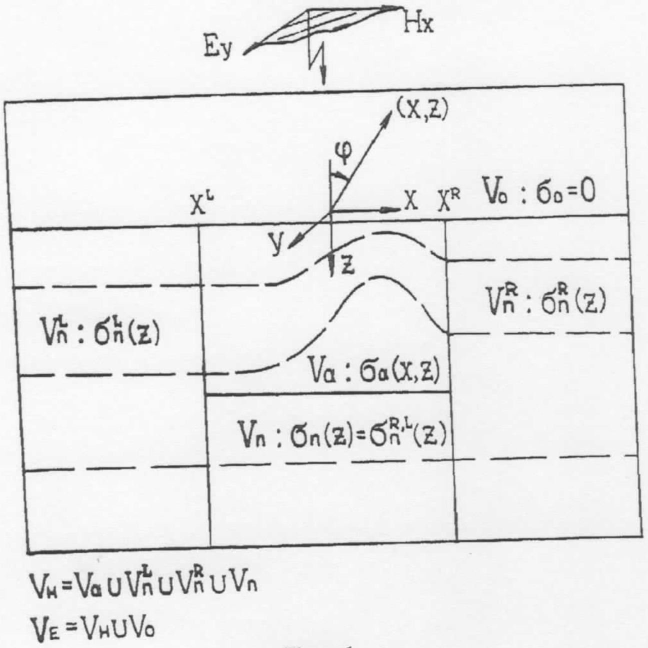

Figure 1

In the case of H-polarization the equation for magnetic field takes the form:

$$\operatorname{div} \frac{1}{K^2} \operatorname{grad} H_y + H_y = 0 \tag{2.3}$$

and the electric field is given by

$$E_x = -\frac{1}{\sigma}\frac{\partial H_y}{\partial z}$$
$$E_z = \frac{1}{\sigma}\frac{\partial H_y}{\partial x}. \tag{2.4}$$

2.2 THE BOUNDARY PROBLEM FORMULATION

We consider the problem of solving equations (2.1) and (2.3) in the bounded rectangular regions V_E (E-polarization) and V_H (H-polarization) with the left, right and lower boundaries lying in the respective parts of the normal section (Fig. 1) and the upper boundary in the atmosphere.

To formulate the boundary problems it is necessary to specify boundary conditions for the electric and magnetic fields. The conditions of the first kind are traditionally used, i.e. the value of electric and magnetic fields are given directly. The boundaries of the region V_E and V_H are assumed to be removed far enough from the region V_a, so that the corresponding anomalous field may be ignored (Pascoe & Jones 1972; Brewitt-Taylor & Weaver 1976; Dmitriev & Barashkov 1979). However, in complicated geoelectric situations it is not clear *a priori* how far should we remove the boundary. The choice of boundary conditions may be incorrect, particularly in an isolator, where the damping of an anomalous field is very slow and determined by the geometry alone. Therefore, in this case the numerical solution of the boundary problem gives rise to great difficulties due to the large dimensions of the modelled region as compared with those of the heterogeneity region V_a.

To overcome these difficulties one may use boundary conditions which either take into account the asymptotic behaviour of anomalous fields far away from the heterogeneities or are based upon integral relationships between the different field components. These ideas in their most general form were expressed by Sveshnikov (1969) and later advanced by Dmitriev (Barashkov 1980) and Weidelt (1975b, 1978). As for 2-D problems of geoelectrics, this approach has been developed most completely by Weaver & Brewitt-Taylor (1976, 1978). In this paper we shall formulate the generalized asymptotic boundary conditions in the atmosphere for the case of *E*-polarization. In the *H*-polarization case the magnetic field is constant. So we have no need of such conditions, and the upper boundary of the region V_H may be coincident with the Earth's surface.

2.3 THE GENERALIZED ASYMPTOTIC BOUNDARY CONDITIONS

To deduce the asymptotic expression for the electric field in air, we use the second of Maxwell's equations and write:

$$E_y(x, z) = i\omega\mu_0 \int_{-\infty}^{x} H_z(x, z)dx + C_1 z + C_0, \qquad z \leqslant 0 \tag{2.5}$$

where C_0 and C_1 are constants.

Let us express the magnetic field $H_z(x, z)$ in (2.5) in terms of its spatial spectrum $h_z(\alpha, z)$:

$$E_y(x, z) = -\frac{\omega\mu_0}{2\pi} \int_{-\infty}^{\infty} \frac{h_z(\alpha, z)}{\alpha} \exp(-i\alpha x)\, d\alpha + C_1 z + C_0, \qquad z \leqslant 0 \tag{2.6}$$

$$h_z(\alpha, z) = \int_{-\infty}^{\infty} H_z(x, z) \exp(i\alpha x)\, dx.$$

It is known that in this model the spectrum $h_z(\alpha, z)$ at any level in the air may be expressed in terms of the one determined at the ground surface:

$$h_z(\alpha, z) = h_z(\alpha, 0) \exp(-|\alpha||z|). \tag{2.7}$$

Substituting (2.7) into (2.6) we obtain:

$$E_y(x, z) = -\frac{\omega\mu_0}{2\pi} \int_{-\infty}^{\infty} h_z(\alpha, 0) \frac{\exp(-|\alpha||z|)}{\alpha} \exp(-i\alpha x)\, d\alpha + C_1 z + C_0. \tag{2.8}$$

The spatial spectrum of the magnetic field H_z at the ground surface may be expanded into the Maclaurin series:

$$h_z(\alpha, 0) = \sum_{P=0}^{\infty} \frac{h_P^+}{P!} \alpha^P, \quad \alpha > 0; \quad h_z(\alpha, 0) = \sum_{P=0}^{\infty} \frac{h_P^-}{P!} \alpha^P, \quad \alpha < 0, \tag{2.9}$$

where h_P are the coefficients of the series. Substituting (2.9) into (2.8) and interchanging the summation and integration we obtain:

$$E_y(x, z) = -\frac{\omega\mu_0}{2\pi} \sum_{P=0}^{\infty} \frac{1}{P!} \int_0^{\infty} \alpha^{P-1} \exp(-|\alpha||z|)\,[\exp(-i\alpha x) h_P^+ - (-1)^P \exp(i\alpha x) h_P^-]\, d\alpha + C_1 z + C_0. \tag{2.10}$$

After the calculation of the tabular integrals in (2.10) (Bateman & Erdelyi 1954) we finally find:

$$E_y = \frac{i\omega\mu_0}{\pi} \left\{ h_0\varphi + \frac{i}{2} \sum_{P=1}^{\infty} \frac{h_P^+ \exp(-iP\varphi) - (-1)^P h_P^- \exp(iP\varphi)}{\rho^P P} \right\} + C_1 z + C_0, \tag{2.11}$$

where φ and ρ are the polar coordinates of the observation point (Fig. 1):

$\rho = \sqrt{x^2 + z^2}, \quad \varphi = \arctan(x/|z|).$

Considering the first $(N+1)$ terms in (2.11) we get the asymptotic expression for the electric field with accuracy $\bar{O}(1/\rho^N)$. The coefficients h_0, C_0 and C_1 in (2.11) are determined by matching this expansion as $x \to \pm\infty$ with the right and left normal electric fields E_y^R and E_y^L, respectively, i.e.

$$h_0 = -H_x^n \frac{Z^R - Z^L}{i\omega\mu_0} = \frac{E_y^R(0) - E_y^L(0)}{i\omega\mu_0}$$

$$C_0 = -H_x^n \frac{Z^R - Z^L}{2} = \frac{E_y^R(0) + E_y^L(0)}{2} \tag{2.12}$$

$$C_1 = -i\omega\mu_0 H_x^n$$

where H_x^n is the normal magnetic field at the ground surface (suitably normalized), $Z^{R,L}$ are the impedances of the right and left normal sections respectively and

$$E_y^{R,L} = -H_x^n (Z^{R,L} + i\omega\mu_0 z), \qquad z \leqslant 0.$$

The difficulty consists in the fact that we do not know the other coefficients h_P in the series (2.11). To exclude the unknown coefficients we construct a differential operator D_N in the form:

$$D_N = L^{(1)} \cdot L^{(2)} \cdot \ldots \cdot L^{(N)} = \prod_{n=1}^{N} L^{(n)}, \tag{2.13}$$

where

$$L^{(n)} = 1 + \frac{\rho}{n} \frac{\partial}{\partial \rho} = 1 + \frac{1}{n} \left(x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} \right).$$

Note that

$$D_N\,[1/\rho^S] = \begin{cases} 1, & S = 0 \\ 0, & 1 \leqslant S \leqslant N \\ (-1)^N C_{S-1}^N / \rho^S, & S > N \end{cases}$$

$(S = 0, 1, 2 \ldots)$,

so an application of operator D_N to the series (2.11) results in the equation:

$$D_N\,[1/\rho^S] = \frac{i\omega\mu_0 h_0}{\pi} \varphi + (N+1)\, C_1 z + C_0 + \bar{O}(1/\rho^N). \tag{2.14}$$

Neglecting the last term in the right side of (2.14) we obtain the approximate equation (written in Cartesian coordinates):

$$\sum_{n=0}^{N} \frac{C_N^n}{n!} \sum_{m=0}^{n} C_n^m x^{n-m} z^m \frac{\partial^n E_y}{\partial x^{n-m} \partial z^m}$$

$$= \frac{i\omega\mu_0 h_0}{\pi} \arctan \frac{x}{|z|} + (N+1) C_1 z + C_0 = f_N(x, z). \tag{2.15}$$

It is valid with accuracy $\bar{O}(1/\rho^N)$ and can be used as the asymptotic boundary condition in the atmosphere.

To realize this condition we have to calculate the derivatives normal to the boundary of the modelled region. It is more convenient however to reduce normal derivatives (beginning from the second order) in the equation (2.15) to the tangential ones using Laplace's equation:

$$\frac{\partial^n E_y}{\partial x^n} = \frac{i^n}{2} \left\{ [1-(-1)^n] \frac{\partial}{\partial x} + [1+(-1)^n] \frac{\partial}{\partial z} \right\} \frac{\partial^{n-1} E_y}{\partial z^{n-1}},$$

$$\frac{\partial^n E_y}{\partial z^n} = \frac{i^n}{2} \left\{ [1+(-1)^n] \frac{\partial}{\partial x} + [1-(-1)^n] \frac{\partial}{\partial z} \right\} \frac{\partial^{n-1} E_y}{\partial x^{n-1}}. \tag{2.16}$$

The substitution of (2.16) into (2.15) produces:

$$E_y + \sum_{n=1}^{N} \frac{C_N^n}{n!} (-\rho)^n \left[\cos n \left(\varphi + \frac{\pi}{2}\right) \frac{\partial}{\partial x} - \sin n \left(\varphi + \frac{\pi}{2}\right) \frac{\partial}{\partial z} \right] \frac{\partial^{n-1} E_y}{\partial x^{n-1}} = f_N(x, z) \tag{2.17}$$

$$E_y + \sum_{n=1}^{N} \frac{C_N^n}{n!} (-\rho)^n \left[-\sin n\varphi \frac{\partial}{\partial x} + \cos n\varphi \frac{\partial}{\partial z} \right] \frac{\partial^{n-1} E_y}{\partial z^{n-1}} = f_N(x, z). \tag{2.18}$$

The conditions (2.17) should be used at the horizontal boundaries while the conditions (2.18) at the vertical boundaries of the region V_E.

In the special case $N = 1$ the relationships (2.17) and (2.18) coincide with the boundary conditions, suggested by Weaver & Brewitt-Taylor. Therefore, these equations may be called the *Weaver–Brewitt-Taylor's generalized asymptotic conditions of order N.*

At the lateral sides of the region V_E, lying in the Earth, the boundary conditions are defined by requiring continuity of the field values from the Earth's surface using the formulae corresponding to the one-dimensional field equations in the laterally homogeneous medium (Weaver & Brewitt-Taylor 1978). At the lower boundary of the region V_E the boundary values are determined, as a rule, by means of the simple interpolation. However, if the bottom of the section is also non-conducting, the boundary conditions may be specified using formulae, analogous to the asymptotic ones written above for the atmosphere (Zhdanov, Varentsov & Golubev 1980).

Experience of the numerical calculations with the asymptotic boundary conditions shows that their application significantly increases the effectiveness of the model.

2.4 THE FINITE-DIFFERENCE APPROXIMATION OF THE BOUNDARY PROBLEM

For the numerical solution of the boundary problem let us introduce the rectangular (uneven) grid $\bar{\Sigma}$ in the regions V_E and V_H.

$$\bar{\Sigma}: \left\{ (x_i z_j) \begin{array}{ll} x_1 = x^-, x_I = x^+, x_i = x_{i-1} + \Delta x_{i-1}, & i = 2, 3, \ldots I \\ z_1 = z^-, z_J = z^+, z_j = z_{j-1} + \Delta z_{j-1} & j = 2, 3, \ldots J. \end{array} \right\}$$

Define on the grid the discrete functions $E(i, j) = E_y(x_i, z_j)$ and $H(i, j) = H_y(x_i, z_j)$. Furthermore, introduce the auxiliary grid $\tilde{\Sigma}$ consisting of the nodes disposed at the centres of grid $\bar{\Sigma}$ cells:

$$\tilde{\Sigma}\left\{(x_{i+1/2}, z_{j+1/2}) \begin{array}{ll} x_{i+1/2} = x_i + \dfrac{\Delta x_i}{2}, & i = 1, \ldots, I-1 \\ z_{j+1/2} = z_j + \dfrac{\Delta z_j}{2} & j = 1, \ldots, J-1 \end{array}\right\}$$

and consider on this grid the discrete function

$$K^2(i \pm 1/2, j \pm 1/2) = K^2(x_{i \pm 1/2}, z_{j \pm 1/2}).$$

To construct a correct difference scheme for solving the boundary problem in the case of E-polarization it is advisable to approximate not the original equation (2.1) but the integral identity resulting from the integration of (2.1) (using the Green theorem) over the elementary cell S_{ij} of the grid $\tilde{\Sigma}$:

$$\int_{L_{ij}} \frac{\partial E_y}{\partial n}\, dl = -\iint_{S_{ij}} K^2 E_y \, dx\, dz, \tag{2.19}$$

where L_{ij} is the rectangular boundary of the cell S_{ij} and n is the outward normal to it. Expressing the integrals in (2.19) approximately in terms of discrete functions $E(i, j)$ and $K^2(i \pm 1/2, j \pm 1/2)$ we obtain the system of difference equations for the electric field values at the interior nodes of the grid $\bar{\Sigma}$:

$$u(i, j) = D_{ij}^{(0)}\,[D_{ij}^{(1)} u(i+1, j) + D_{ij}^{(2)} u(i, j+1) + D_{ij}^{(3)} u(i-1, j) + D_{ij}^{(4)} u(i, j-1)] \tag{2.20}$$

$$i = 2, \ldots, I-1; \qquad j = 2, \ldots, J-1$$

where $u(i, j) = E(i, j)$

$$D_{ij}^{(1)} = \frac{1}{\Delta x_i \tilde{\Delta x_i}} \qquad D_{ij}^{(3)} = \frac{1}{\Delta x_{i-1} \tilde{\Delta x_i}}$$

$$D_{ij}^{(2)} = \frac{1}{\Delta z_j \tilde{\Delta z_j}} \qquad D_{ij}^{(4)} = \frac{1}{\Delta z_{j-1} \tilde{\Delta z_j}}$$

$$D_{ij}^{(0)} = \left[\sum_{l=1}^{4} D_{ij}^{(l)} - \frac{1}{4S_{ij}} \sum_{p=0}^{1} \sum_{q=0}^{1} K_{p,q}^2 S^{p,q}\right]^{-1}$$

$$\tilde{\Delta x_i} = (\Delta x_{i-1} + \Delta x_i)/2 \qquad \tilde{\Delta z_j} = (\Delta z_{j-1} + \Delta z_j)/2$$

$$K_{p,q}^2 = K^2(i + p - 1/2, j + q - 1/2), \qquad S^{p,q} = \Delta x_{i+p-1} \Delta z_{j+q-1}.$$

In the case of H-polarization, approximating the integral identity, resulting from the equation (2.3),

$$\int_{L_{ij}} \frac{1}{K^2} \frac{\partial H_y}{\partial n}\, dl = -\iint_{S_{ij}} H_y \, dx dz \tag{2.22}$$

we obtain the system of difference equations of general type (2.20) for the magnetic field ($u(i, j) = H_y(i, j)$) with coefficients:

$$D_{ij}^{(1)} = \frac{\sum_{q=0}^{1} S^{1,q}/K_{1,q}^2}{2S_{ij}(\Delta x_i)^2}, \qquad D_{ij}^{(3)} = \frac{\sum_{q=0}^{1} S^{0,q}/K_{0,q}^2}{2S_{ij}(\Delta x_{i-1})^2}$$

$$D_{ij}^{(2)} = \frac{\sum_{p=0}^{1} S^{p,1}/K_{p,1}^2}{2S_{ij}(\Delta z_j)^2}, \qquad D_{ij}^{(4)} = \frac{\sum_{q=0}^{1} S^{p,0}/K_{p,0}^2}{2S_{ij}(\Delta z_{j-1})^2} \tag{2.23}$$

$$D_{ij}^{(0)} = \left[\sum_{l=1}^{4} D_{ij}^{(l)} - 1\right]^{-1}.$$

Thus, we have constructed a finite-difference approximation of this boundary value problem at the interior nodes of the grid $\bar{\Sigma}$. Now we consider an approximation for boundary conditions themselves.

The approximation of the conditions of the first kind does not lead to any difficulties. The field values at the boundary nodes of the grid are simply made equal to the normal fields values. In this case we obtain the homogeneous 5-point difference schemes of the type (2.20) for both field polarizations.

The application of the generalized asymptotic boundary conditions results in additional difference equations at the boundary nodes of the grid $\bar{\Sigma}$ (Varentsov & Golubev 1980b), distorting the homogeneity of the difference scheme. In the general case (excluding the conditions of first order) we do not obtain the usual 5-point schemes.

2.5 THE SOLUTION OF THE SYSTEM OF FINITE-DIFFERENCE EQUATIONS USING THE SUCCESSIVE OVER-RELAXATION TECHNIQUE

The systems of finite-difference equations, obtained earlier, may be written in a matrix form:

$$A \cdot U = C \tag{2.24}$$

where U is the column of unknown values of electric or magnetic field at the grid nodes, A is the matrix of the system, C is a column vector containing boundary terms. The structure of the matrix A depends considerably on the way the elements of the unknown values vector U are ordered and on the choice of boundary conditions.

In the simplest case, when using the boundary conditions of the first kind, the matrix A has a 5-diagonal structure. It is possible to solve such a system by the direct methods of linear algebra (Greenfield 1965; Brewitt-Taylor & Weaver 1976; Madden & Swift 1972). Iterative methods require comparatively less computer storage and are more stable with respect to the rounding errors (Forsythe & Wasow 1960). One of the most effective among them (for the problem under consideration) is the successive over relaxation (SOR) technique (Forsythe & Wasow 1960). This technique was applied by Tatrallyay (1978a), Müller & Losecke (1975) and Varentsov & Golubev (1980a). The corresponding iteration procedure for solving the system (1.20) is as follows:

$$U^{(t+1)}(i,j) = (1-\nu)\, U^{(t)}(i,j) + \nu D_{ij}^{(0)}\, \{D_{ij}^{(1)}\, U^{(t)}(i+1,j) + D_{i,j}^{(2)}\, U^{(t)}(i,j+1)\, D_{ij}^{(3)}\, U^{(t+1)}(i-1,j) + D_{ij}^{(4)}\, U^{(t+1)}(i,j-1)\} \tag{2.25}$$

$$i = 2, 3, \ldots, I-1; \qquad j = 2, 3, \ldots, J-1$$

where t is the iteration serial number and $\nu\,(\geqslant 1)$ is the relaxation factor determining the rapidity of convergence of method considered.

In the matrix form the formulae (2.25) are written as

$$U^{(t+1)} = R(\nu)\, U^{(t)} \tag{2.26}$$

where $R(\nu)$ is the transition matrix. It also connects the increments of calculated field values $\Delta U^{(t)} = U^{(t+1)} - U^{(t)}$ for two successive iterations

$$\Delta U^{(t+1)} = R(\nu)\cdot \Delta U^{(t)}. \tag{2.27}$$

The most important question which arises in realizing the SOR method concerns with the search for the relaxation factor ν_{opt} the correct choice of which results in a considerable decrease of the number of iterations required (Forsythe & Wasow 1960; Tatrallyay 1978a). In the general case it is difficult to give *a priori* an effective evaluations of the optimal value of the relaxation factor. It is better to determine it during the iteration procedure, using the Young-Frankel theory (Forsythe & Wasow 1960). In the framework of this theory the relaxation factor ν_{opt} is predicted using the maximum λ_m of absolute eigenvalues of the transition matrix $R(\nu)$:

$$\nu_{\text{opt}} = 2\left[1 + \sqrt{1 - (\lambda_m + \nu_0 - 1)^2/(\lambda_m \cdot \nu_0^2)}\right]^{-1} \tag{2.28}$$

where ν is the initial value of relaxation factor ($\nu_0 < \nu_{\text{opt}}$, for example, $\nu = 1$). The value λ_m is estimated after a certain initial interval N_0 of the iteration process with a constant relaxation factor ν_0:

$$\lambda_m \simeq \frac{1}{N_0 - 1} \sum_{t=1}^{N_0 - 1} \frac{\| \Delta U^{(t+1)} \|}{\| \Delta U^{(t)} \|} \tag{2.29}$$

where

$$\| \Delta U^{(t)} \| = \max_{i,j} | U^{(t+1)}(i,j) - U^{(t)}(i,j) |$$

$$i = 1, 2, \ldots, I; \qquad j = 1, 2, \ldots, J.$$

Substituting the approximate value λ_m in (2.28), we obtain an estimate (*prediction*) of the optimal relaxation factor ν_{opt}. Such an approach will give ν within a small vicinity of ν_{opt} after few iterations. It is possible to improve the estimate of ν_{opt} using a special *correction* procedure (Varentsov & Golubev 1980a). The latter is based on the scanning of relaxation factors close to ν_{opt} and choosing the one which provides the most rapid convergence. It is advisable to fulfill such a correction periodically, and in this case we can obtain even better convergence than that in the case of an optimal but constant relaxation factor.

When we use the asymptotic boundary conditions it is worthwhile applying the SOR modification with two relaxation factors – for interior and boundary nodes – due to the different types of equations describing the field within the region and the boundary conditions. To optimize the 'interior' relaxation factor ν^i the scheme, described above, can be applied. The 'exterior' relaxation factor ν^e is determined by means of a simple algorithm: if the maximum of $|\Delta U^{(t)}|$ is achieved at the boundary, then the value ν^e is decreased stepwise; to the contrary, it remains unchanged. Practical computations show that ν^e is always less than 1, therefore, at the beginning of the iteration we can assign $\nu_0^e = 1$.

In the conclusion of this section we note that the number of iterations required to solve the linear system with a given accuracy depends not only on the rapidity with which the iteration process converges, but also on a suitable choice of the initial approximation $U^{(0)}$. It is usually chosen by means of the horizontal interpolation, i.e. wholly determined by the conductivity of normal sections $\sigma^{R,L}(z)$. For constructing more accurate initial approximation it is necessary to take into account the anomalous structure of the geoelectric section. Since the field computation is usually carried out for monotonic sequence of periods, the initial approximation is naturally assigned by some transformation (for example, linear – Müller & Losecke 1975; Varentsov & Golubev 1980a) of the solution obtained for the previous period. It is also desirable to construct the initial approximation by means of interpolation of the field values determined on the smaller grid (Müller & Losecke 1975; Marchuk & Shaidurov 1979).

2.6 TESTING OF THE MODEL VALIDITY

One of the most important questions in the numerical modelling is the testing of the validity of the modelled field. The modelling errors are caused in general by three main reasons: (1) the errors connected with the grid approximation of the boundary problem, (2) the errors arising under the solution of the system of finite-difference equations, (3) the errors resulting from the numerical differentiation of the components of the field, not occurring in the difference scheme (the magnetic field in the case of E-polarization and the electric field in the case of H-polarization).

In the majority of the works which have been devoted to the finite-difference modelling attention is paid mainly to the errors of the second type, i.e. arising in solving the linear system of equations (Pascoe & Jones 1972; Müller & Losecke 1975; Brewitt-Taylor & Weaver 1976). In this case testing of the solution validity is fulfilled using the following criterion:

$$\frac{\| \Delta U^{(t)} \|}{|Z^{\mathrm{R}} + Z^{\mathrm{L}}|/2} < \epsilon_1 \ (E\text{-polarization})$$

$$\| \Delta U^{(t)} \| < \epsilon_1 \ (H\text{-polarization}) \tag{2.30}$$

where ϵ_1 is the required accuracy and the norm is determined in accordance with (2.29).

The errors arising in numerical differentiation (the errors of the third type) can be estimated by means of comparing the results obtained using the different numerical techniques (Varentsov & Golubev 1980a). Müller & Losecke (1975) have suggested a criterion which simultaneously tests the errors of second and third types. It is based on the analysis of the increment norm of the impedances $Z^{(t)}$ calculated in each m iterations at the ground surface:

$$\frac{1}{m} \frac{\| Z^{(t+m)} - Z^{(t)} \|}{\| Z^{(t+m)} \|} < \epsilon_2. \tag{2.31}$$

It should be noted that the criteria, cited above, do not completely evaluate the real modelling accuracy, because errors may arise at the stage of the grid approximation (Tatrallyay 1978a). Therefore, it is of great importance to have a method to test the total error of the model. One of the ways to make such estimates is an analysis of the functional relationships between different field components, for example, the Hilbert–Kertz relationship:

$$H_x^a(x, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H_z^a(x^1, 0)}{x^1 - x}\, dx^1$$

$$H_z^a(x, 0) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H_x^a(x^1, 0)}{x^1 - x}\, dx^1. \tag{2.32}$$

The appropriate criterion may be based on the validity of (2.32) when substituting components H_x^a and H_z^a determined from a finite-difference solution (Varentsov & Golubev 1980a). This criterion is not universal, however, as it does not explicitly take into account the conductivity structure of the model.

The other criterion, based on the integral relationships between the fields in the region of anomalous conductivity and those at the Earth's surface is significantly more effective. Consider, for example, the case of E-polarization, when the parameters of the right and the left normal sections coincide ($\sigma^L(z) = \sigma^R(z) = \sigma_n(z)$):

$$E_y(x, 0) = E_y^n + i\omega\mu_0 \iint_{Va} G_n(x, 0; \xi, \zeta) \, [\sigma_a(\xi, \zeta) - \sigma_n(\zeta)] \, E_y(\xi, \zeta) d\xi \, d\zeta \qquad (2.33)$$

where E_y^n is the normal electric field at the Earth's surface, G_n is the Green function for the normal section. Substituting into the right side of (2.33) the values of electric field, determined via the finite-difference scheme, integrating numerically and comparing the result E_y^{int} with the finite-difference solution E_y at the Earth's surface, we obtain the required criterion:

$$\frac{\| E_y^{\text{int}} - E_y \|}{\| E_y \|} \leqslant \epsilon_3 . \qquad (2.34)$$

Essentially, this criterion provides the coincidence of the finite-difference modelling result and the solution by means of integral equations technique (Berdichevsky & Dmitriev 1976; Weidelt 1978).

2.7 NUMERICAL EXAMPLES

The algorithm described above is realized in the program package FDM. To illustrate the effectiveness of these programs consider the models in Dmitriev, Zakharov & Kokotushkin (1973). For the case of E-polarization let us take a model consisting of a rectangular subsurface conducting body buried in the three-layered normal section (Fig. 2). The apparent resistivity curves, calculated by the integral equations technique (the thick line) and the results of finite-difference computations are both shown in Fig. 2 (the circles denote the use of the boundary conditions of the first kind, the crosses denote the asymptotic boundary conditions of the first order, the numbers at the curves specify the distance (in km) from the sounding point to the centre of the heterogeneity, the dotted lines correspond to 1-D curves in the centre of the model and out of the heterogeneity, the model parameters are given in $S \cdot m^{-1}$ and km).

The problem was solved using ordinary boundary conditions in the region V_E with sizes 5400 x 540 km on the grid of (37 x 24) nodes. In the case of asymptotic conditions the sizes of the grid were diminished to 240 x 170 km and the number of nodes was (39 x 35).

As it is seen, the results obtained by finite-difference modelling and by the integral equations technique match well enough the best accuracy being reached with the asymptotic boundary conditions.

The examples of calculations for the model of a horst in the case of H-polarization are shown in Fig. 3. The continuous lines denote the apparent resistivity curves obtained by the integral equations technique and the circles denote the results of finite-difference modelling. Here also good agreement is observed between the curves obtained using different techniques.

Some discrepancies in Figs 2 and 3 may be due to the approximate description of the perfect isolators and conductors (the real conductivities of models are given in Figs 2 and 3

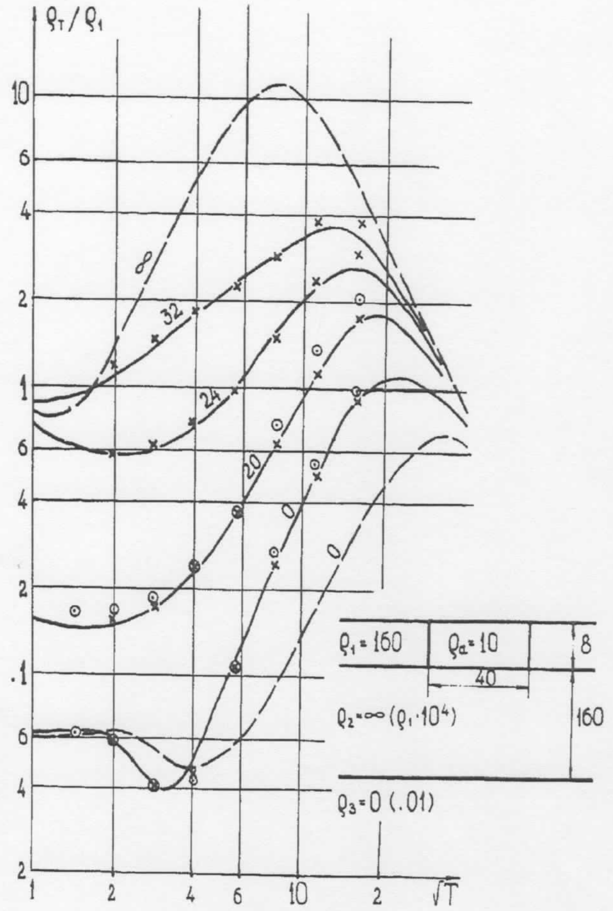

Figure 2

in parentheses) and to the continuous approximation of the contact between the surrounding medium and the inhomogeneity in the finite-difference method.

3 Three-dimensional modelling

3.1 THE GEOELECTRIC MODEL AND FIELD EQUATIONS

Consider the geoelectric model, presented in Fig. 4, in which the three-dimensional inhomogeneous region V_a in the form of a rectangular prism (elongated to infinity in the y-axis) with an anomalous conductivity distribution $\sigma_a(x, y, z)$ is buried in the horizontally layered medium. We suppose that some distance along the y-axis the three-dimensional heterogeneity becomes two-dimensional, i.e.

$$\sigma_a(x, y, z) = \sigma_a(x, z) \text{ for } y \geqslant y^+, y \leqslant y^-.$$

As in the two-dimensional model (Fig. 1), the region V_a is surrounded by three domains of normal section: the left V_n^{L}, the right V_n^{R} and the lower V_n – all having one-dimensional conductivity distribution.

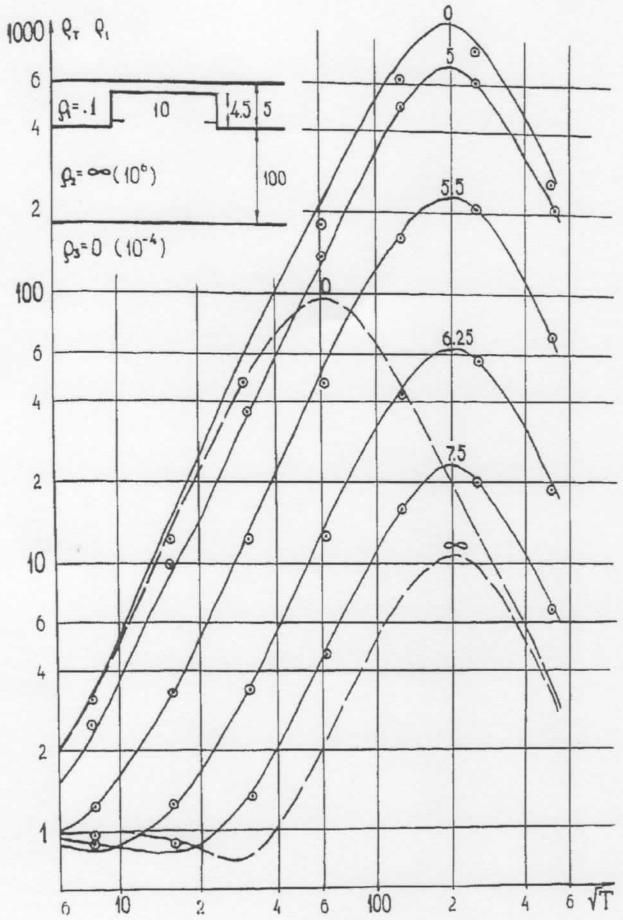

Figure 3

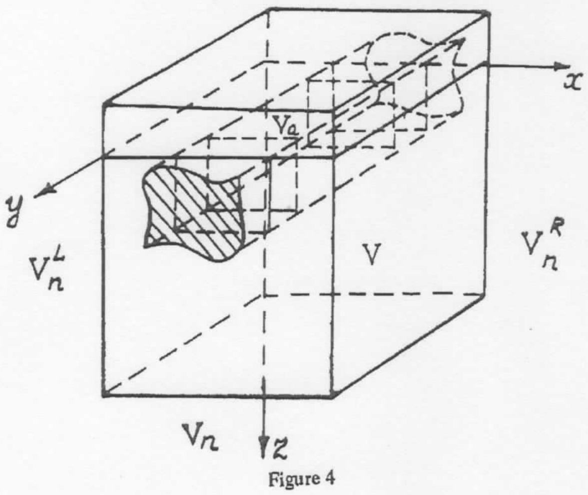

Figure 4

We assume, as in Section 2, that $\mu \equiv \mu_0$. Neglecting the displacement currents we may write:

$$\Delta \mathrm{E} - \mathrm{grad}\,(\mathrm{div}\,\mathrm{E}) + K^2 \mathrm{E} = 0. \tag{3.1}$$

The magnetic field is expressed through the electric field using the Maxwell's second equation:

$$\mathrm{H} = \frac{1}{i\omega\mu_0}\,\mathrm{rot}\,\mathrm{E}. \tag{3.2}$$

Taking into account that

$$\mathrm{div}\,\mathrm{E} = -\,\mathrm{E}\,\frac{\mathrm{grad}\,\sigma}{\sigma}\quad(\sigma \neq 0)$$

(3.1) may be rewritten in the form:

$$\Delta \mathrm{E} + \mathrm{grad}\left(\mathrm{E}, \frac{\mathrm{grad}\,\sigma}{\sigma}\right) + K^2 \mathrm{E} = 0. \tag{3.3}$$

The use of (3.3) instead of (3.1) in the numerical solution of this problem has several advantages. First, the finite-difference approximation of equation (3.3) can be accomplished by a 7-point scheme, which diminishes the computer storage required and the computational time. Secondly, in approximating the equation (3.3) in the regions where grad $\sigma = 0$ the second term of the equation vanishes while in approximating the equation (3.1) the corresponding term remains in the regions with grad $\sigma \neq 0$ as well as in the regions, where grad $\sigma = 0$, which leads to the additional errors in the calculations (Zhdanov & Spichak 1980).

Let us consider the scattering of the arbitrarily polarized plane electromagnetic wave by the three-dimensional heterogeneities, described above. The normal fields in the right and left parts of the normal section are defined as follows:

$$\mathrm{E}^{\mathrm{R,L}}(z) = \mathrm{P}_{\mathrm{H}^n}(Z^{\mathrm{R,L}} + i\omega\mu_0 z) \tag{3.4}$$

where $\mathrm{P}_{\mathrm{H}}{}^n = \{H_y^n, -H_x^n, 0\}$ is the vector of normal magnetic field components at the ground surface and $Z^{\mathrm{R,L}}$ are the impedances of normal sections.

3.2 BOUNDARY CONDITIONS

In the three-dimensional case we shall utilize, as in Section 2, the asymptotic boundary conditions, which diminish significantly the region V where the solution is required.

Let us express the electric field in the air in the form:

$$E_{x,y} = \tilde{E}_{x,y} + C_{0x,y} + C_{1x,y}z + C_{2x,y}\arctan\frac{x}{|z|}\quad(z \leqslant 0) \tag{3.5}$$

where $\tilde{E}_{x,y}$ are the components of the electric field vanishing at infinity and $C_{0x,y}$, $C_{1x,y}$, $C_{2x,y}$ are constants determined from the condition:

$$E_{x,y}\big|_{x=\pm\infty} = E_{x,y}^{\mathrm{R,L}}. \tag{3.6}$$

Using the second and third of Maxwell's equations, we write:

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_0 H_z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{\partial E_z}{\partial z}. \tag{3.7}$$

Substituting (3.5) into (3.7), introducing the spatial spectra of the electromagnetic field

$$\left\{\begin{matrix} \mathrm{e}(\alpha,\beta,z) \\ \mathrm{h}(\alpha,\beta,z) \end{matrix}\right\} = \iint_{-\infty}^{\infty} \left\{\begin{matrix} \mathrm{E}(x,y,z) \\ \mathrm{H}(x,y,z) \end{matrix}\right\} \exp\left[i(\alpha x + \beta y)\right] dxdy \tag{3.8}$$

and taking into account that everywhere in the atmosphere

$$\begin{aligned} &h_z(\alpha,\beta,z) = h_z(\alpha,\beta,0)\, e^{n_0 z} \\ &e_z'(\alpha,\beta,z) = e_z'(\alpha,\beta,0)\, e^{n_0 z}, \; z \leqslant 0 \\ &n_0 = \sqrt{\alpha^2+\beta^2}, \; e_z'(\alpha,\beta,z) = \frac{\partial}{\partial z} e_z(\alpha,\beta,z) \end{aligned} \tag{3.9}$$

we obtain

$$\left\{\begin{matrix} \tilde{e}_x \\ \tilde{e}_y \end{matrix}\right\} = \left\{\begin{matrix} e_x^{\mathrm{TE}} + e_x^{\mathrm{TM}} \\ e_y^{\mathrm{TE}} + e_y^{\mathrm{TM}} \end{matrix}\right\} + 2\pi^2 \frac{\exp(|\alpha| z)}{n_0^2}\, \delta(\beta) \left\{\begin{matrix} -i\alpha C_{2x} + i\beta C_{2y} \\ -i\beta C_{2x} - i\alpha C_{2y} \end{matrix}\right\} \tag{3.10}$$

where

$$\begin{aligned} &\left\{\begin{matrix} e_x^{\mathrm{TE}} \\ e_y^{\mathrm{TE}} \end{matrix}\right\} = \frac{i\omega\mu_0 e^{n_0 z}}{n_0^2} \left\{\begin{matrix} -i\beta \\ i\alpha \end{matrix}\right\} h_z(\alpha,\beta,0) \\ &\left\{\begin{matrix} e_x^{\mathrm{TM}} \\ e_y^{\mathrm{TM}} \end{matrix}\right\} = \frac{e^{n_0 z}}{n_0^2} \left\{\begin{matrix} -i\alpha \\ -i\beta \end{matrix}\right\} e_z'(\alpha,\beta,0) \end{aligned} \tag{3.11}$$

and $\delta(\beta)$ is a delta-function.

The formulae (3.11) justify the introduction of the indices TE and TM, since they show that $e_{x,y}^{\mathrm{TE}}$ and $e_{x,y}^{\mathrm{TM}}$ are the spectra of transverse-electric ($E_{x,y}^{\mathrm{TE}}$) and transverse-magnetic ($E_{x,y}^{\mathrm{TM}}$) modes of the field $\tilde{E}_{x,y}$. Thus, the formulae (3.5) can be written in the following manner:

$$E_{x,y} = E_{x,y}^{\mathrm{TE}} + E_{x,y}^{\mathrm{TM}} + C_{0x,y} + C_{1x,y} z + C_{2x,y} \arctan \frac{x}{|z|}. \tag{3.12}$$

Notice also that the vertical component of the electric field belongs to transverse-magnetic mode only:

$$E_z = E_z^{\mathrm{TM}} \tag{3.13}$$

$$e_z = e_z^{\mathrm{TM}}. \tag{3.14}$$

In the equations (3.12) and (3.13) we have:

$$E_{x,y,z}^{\mathrm{TE,TM}} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} e_{x,y,z}^{\mathrm{TE,TM}} \exp\left[-i(\alpha x + \beta y)\right] d\alpha\, d\beta \tag{3.15}$$

and, in a similar way to the two-dimensional case, we expand the spectra $h_z(\alpha,\beta,0)$ and $e_z'(\alpha,\beta,0)$ in Maclaurin's series:

$$\left\{\begin{matrix} h_z(\alpha,\beta,0) \\ e_z'(\alpha,\beta,0) \end{matrix}\right\} = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} C_l^m \left\{\begin{matrix} h_{lm} \\ e_{lm} \end{matrix}\right\} (-i\alpha)^m (-i\beta)^{l-m}. \tag{3.16}$$

Substituting (3.11) and (3.14) into (3.15) and taking into account (3.16) we find:

$$\begin{Bmatrix} E_x^{TE} \\ E_y^{TE} \end{Bmatrix} = \frac{i\omega\mu_0}{2\pi} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} C_l^m h_{lm} \frac{\partial^l}{\partial x^m \partial y^{l-m}} \begin{Bmatrix} \dfrac{\partial}{\partial y} \\ -\dfrac{\partial}{\partial x} \end{Bmatrix} \ln(z+R) \qquad (3.17)$$

$$\begin{Bmatrix} E_x^{TM} \\ E_y^{TM} \\ E_z^{TM} \end{Bmatrix} = \frac{1}{2\pi} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} C_l^m e_{lm} \frac{\partial^l}{\partial x^m \partial y^{l-m}} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} \ln(z+R)$$

where $R = \sqrt{x^2+y^2+z^2}$.

To deduce the formulae (3.17) we used the Fourier transform:

$$\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \frac{e^{n_0 z}}{n_0^2} \exp[-i(\alpha x + \beta y)]\, d\alpha d\beta = \frac{1}{2\pi} \ln(z+R).$$

The substitution of (3.17) into (3.12) produces the asymptotic expression for the electric fields in the air for large R. However, the coefficients h_{l_m} and e_{l_m} in these expressions remain undefined. To exclude these unknown coefficients one may use the procedure, described in Section 2 for the two-dimensional case. For the construction of three-dimensional analogues of the Weaver–Brewitt-Taylor conditions (of the first order) let us consider the operator (Varentsov & Golubev 1980b):

$$D_1 = 1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \qquad (3.18)$$

Substituting (3.17) into (3.12), (3.13) and using (3.18) we obtain:

$$\left(1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\mathbf{E} = \left(1 + R\frac{\partial}{\partial R}\right)\mathbf{E}$$

$$= \left(\frac{Z^R - Z^L}{\pi}\theta + \frac{Z^R + Z^L}{2} + i\omega\mu_0 z\right) \mathrm{P}_{Hn} + \bar{O}(1/R), \theta = \arctan(x/|z|). \qquad (3.19)$$

The relationship (3.19) is the three-dimensional analogy of the asymptotic boundary conditions of Weaver and Brewitt-Taylor.

It should be noted that using the relationships (3.17) we may construct as in the two-dimensional case the asymptotic boundary conditions of the arbitrary order N.

3.3 THE FINITE-DIFFERENCE APPROXIMATION OF THE 3-D PROBLEM AND THE METHOD OF ITS SOLUTION

To solve the equation (3.3) numerically we subdivide the region V into the elementary rectangular prisms by some three-dimensional grid $\bar{\Sigma}$ and introduce a second grid $\tilde{\Sigma}$, the nodes of which are situated in the centres of the corresponding cells of the grid $\bar{\Sigma}$. The node

(l, m, n) of the grid $\bar{\Sigma}$ corresponds to the point $(x_l, y_m, z_n)(1 \leq l \leq L, 1 \leq m \leq M, 1 \leq n \leq N)$. The electric field E is denoted at the nodes of the grid $\bar{\Sigma}$ by $\mathbf{U}(l, m, n) = \mathbf{E}(x_l, y_m, z_n)$, and the wavenumber $K^2(l \pm \frac{1}{2}, m \pm \frac{1}{2}, n \pm \frac{1}{2})$ at the nodes of the auxiliary grid $\tilde{\Sigma}$. To obtain a difference approximation of equation (3.3) let us apply as in the two-dimensional case the integro-interpolation technique (the balance method). Integrating (3.3) over the elementary cell V_{lmn} of the grid $\tilde{\Sigma}$, we have:

$$\iiint_{V_{lmn}} \Delta \mathbf{E}\, dV + \iiint_{V_{lmn}} \mathrm{grad}\left(\mathbf{E}, \frac{\mathrm{grad}\,\sigma}{\sigma}\right) dV + \iiint_{V_{lmn}} K^2 \mathbf{E}\, dV = 0. \quad (3.20)$$

Applying Gauss's theorem to the first and second integrals in (3.20) and replacing σ by K^2 we obtain:

$$\iint_{S_{lmn}} \frac{\partial \mathbf{E}}{\partial n} dS + \iint_{S_{lmn}} \left(\mathbf{E}, \frac{\mathrm{grad}\, K^2}{K^2}\right) d\mathbf{S} + \iiint_{V_{lmn}} K^2 \mathbf{E}\, dV = 0. \quad (3.21)$$

This vectorial equation is equivalent to three scalar equations, differing from the two-dimensional equation (2.19) only by the presence of the term with grad K^2 and permits a simple finite-difference approximation, resulting in a 7-point difference scheme (Zhdanov & Spichak 1980):

$$\begin{aligned} \mathbf{U}(l, m, n) = \hat{D}^{(0)}_{lmn} \{ & \hat{D}^{(1)}_{lmn} \mathbf{U}(l+1, m, n) + \hat{D}^{(2)}_{lmn} \mathbf{U}(l, m+1, n) \\ & + \hat{D}^{(3)}_{lmn} \mathbf{U}(l, m, n+1) + \hat{D}^{(4)}_{lmn} \mathbf{U}(l-1, m, n) + \hat{D}^{(5)}_{lmn} \mathbf{U}(l, m-1, n) \\ & + \hat{D}^{(6)}_{lmn} \mathbf{U}(l, m, n-1) \} \end{aligned} \quad (3.22)$$

$$l = 2, \ldots, L-1; \qquad m = 2, \ldots, M-1; \qquad n = 2, \ldots, N-1,$$

where $\hat{D}^{(i)}_{lmn}$ $(i = 0, 1, \ldots 6)$ are the matrix coefficients (of the third order) determined by the grid geometry and the discrete function $K^2(l \pm \frac{1}{2}, m \pm \frac{1}{2}, n \pm \frac{1}{2})$.

To solve the system (3.22) one may use the SOR technique. However, due to the vectorial character of the difference scheme it is advisable to apply the blocked modification of this technique (SBOR) each block consisting of three electric field components at one node (Zhdanov & Spichak 1980):

$$\begin{aligned} \mathbf{U}^{(t+1)}(l, m, n) = (1-\nu)\, \mathbf{U}^{(t)}(l, m, n) + \nu \hat{D}^{(0)}_{lmn} \{ & \hat{D}^{(1)}_{lmn} \mathbf{U}^{(t)}(l+1, m, n) \\ & + \hat{D}^{(2)}_{lmn} \mathbf{U}^{(t)}(l, m+1, n) + \hat{D}^{(3)}_{lmn} \mathbf{U}^{(t)}(l, m, n+1) \\ & + \hat{D}^{(4)}_{lmn} \mathbf{U}^{(t+1)}(l-1, m, n) + \hat{D}^{(5)}_{lmn} \mathbf{U}^{(t+1)}(l, m-1, n) \\ & + \hat{D}^{(6)} \mathbf{U}^{(t+1)}(l, m, n-1) \} \end{aligned} \quad (3.23)$$

$$l = 2, \ldots, L-1; \qquad m = 2, \ldots, M-1; \qquad n = 2, \ldots, N-1.$$

The relaxation factor ν is chosen using the scheme suggested for the two-dimensional case.

3.4 THE ESTIMATION OF THE VALIDITY

The validity testing in the three-dimensional should be carried out in accordance with the same principles as formulated in Section 2.6. The criteria (2.30), (2.31) may be used in the three-dimensional case without any changes, but the criteria based on the Hilbert–Kertz relationships and on the integral relation between the fields in the anomalous region and those at the ground surface require an appropriate modification. For example, the following relationship is the three-dimensional analogue of the formula (2.35) (Weidelt 1975a):

$$\mathbf{E}(\mathbf{r}_0) = \mathbf{E}^n(\mathbf{r}^0) + i\omega\mu_0 \iiint_{V_a} \hat{G}(\mathbf{r}, \mathbf{r}_0)\, [\sigma_a(\mathbf{r}) - \sigma_n(\mathbf{r})]\, \mathbf{E}(\mathbf{r})\, dV. \tag{3.24}$$

Here the parameters of the right, left and lower normal sections are assumed to be equal: $\sigma^L(z) \equiv \sigma^R(z) = \sigma_n(z)$; $\hat{G}$ is the Green tensor for the normal section and $\mathbf{r}_0$ – the observation point which lies on the ground surface.

Substituting the calculated values of the electromagnetic field into the integral on the right side of (3.24), integrating numerically and comparing the result $\mathbf{E}^{int}$ with the finite-difference solution $\mathbf{E}$ we can estimate three-dimensional modelling validity (in accordance with (2.34)).

4 Conclusion

In describing these algorithms for finite-difference modelling we have not considered a many of the important computational, technological and methodological questions concerning the practical realization of this approach. It seems that it is considerably more important to formulate the main ideas of the construction of the effective algorithms instead of describing technical details. The problem of realizing these algorithms in concrete computer programs is of great importance, but it goes beyond the framework of this paper. Moreover, we have only shown some test models although our two-dimensional programs are widely used for modelling real geoelectrical situations.

The main prospects (and at the same time difficulties) of the further development of the finite-difference approach are connected with the increase of the modelling effectiveness (mainly, in the three-dimensional case). We hope that the ideas and principles formulated above will produce progress in solving this problem.

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