3D ELECTROMAGNETIC INVERSION USING INTEGRAL EQUATIONS

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ABSTRACT


Utilizing electromagnetic data in geophysical exploration work is difficult when measured responses are complicated by the effects of 3D structures. 1D and 2D models may not be capable of accurately simulating the physical processes that contribute to a measured response. 3D conductive-host modelling is difficult, costly and time-consuming. Using a 3D inverse procedure it is possible to automate the interpretation of controlled-source electromagnetic data. This procedure uses an inverse formulation based on frequency-domain, volume integral equations and a pulse-basis representation for the internal electrical field and anomalous conductivity. Beginning with an initial model composed of a 3D inhomogeneous region residing in a laterally homogeneous (layered-earth) geoelectrical section, iterative least-squares algorithms are used to refine the geometry and the conductivity of the inhomogeneity. This novel approach for 3D electromagnetic interpretation yields a reliable and stable inverse solution provided constraints on how much the variable can change at each iteration are incorporated. Integral-equation-based inverse formulations that do not correctly address the non-linearity of this inverse problem may have poor convergence properties, particularly when dealing with the high conductivity contrasts that are typical of many exploration problems.

While problems associated with contamination of the data by random noise and non-uniqueness of solutions do not usually influence the inverse solution in an adverse manner, problems associated with model inadequacy and errors in an assumed background conductivity structure can produce undesirable effects.

INTRODUCTION

A procedure for inverting three-dimensional (3D) electromagnetic data is described. The data are assumed to be the product of a controlled source (loop or grounded-wire) survey over a 3D inhomogeneity embedded in a layered earth. This assump-

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tion is a consequence of the limitations of the forward modelling algorithm used to compute model responses.


The approach used here to solve the 3D electromagnetic inverse problem is to start with a good initial model obtained using an imaging technique for electromagnetic sounding data. Next the geometry of the 3D inhomogeneous region is refined, in regard to both its size and shape, using a singular value decomposition and a penalized least-squares formulation. Finally a damped or smoothed least-squares formulation is used to refine conductivity estimates within the inhomogeneity.

For efficiency, forward and inverse models for the least-squares solutions are computed in the frequency domain. Multi-frequency in-phase and quadrature components are inverted simultaneously, as are all of the responses measured for all of the loop or grounded-wire sources in the vicinity of the inhomogeneity. Details of the forward and inverse solutions are presented together with examples that illustrate some of the strengths and weaknesses of this procedure. Additional information and examples on this approach for 3D electromagnetic inversion are provided in Eaton (1987).

**INTEGRAL-EQUATION SOLUTION FOR THE FORWARD PROBLEM**

The electromagnetic response of the 3D model illustrated in Fig. 1 is given by:

\[ F(r, \omega) = F^0(r, \omega) + \int_{V'} \sigma_a(r') G^i(r, r'; \omega) E(r', \omega) \, \, dr', \]

where \( F \) is either the electric (\( E \)) or magnetic (\( H \)) field at a position \( r \). A loop or grounded-wire source, oscillating at the surface of the model with an angular frequency \( \omega \), produces a primary (layered-earth) field \( F^0 \) in the absence of the 3D inhomogeneity (\( V' \)). The anomalous conductivity (\( \sigma_a \)) of the body is the difference between its actual conductivity and the conductivity of the layer or layers in which it resides. \( G^i \) is either the electric-field (\( G^e \)) or magnetic-field (\( G^m \)) Green’s tensor relating the layered-earth field at \( r \) to a dipole current source at \( r' \). Expressions for the elements of each tensor are given by Wannamaker, Hohmann and San Filipo (1984).

A method for estimating the response of a model given a distribution of anomalous conductivities, i.e. the forward solution, is described in detail by Newman, Hohmann and Anderson (1986) and consists of two steps. First one solves for the electric field in the inhomogeneous region (the internal field) using a pulse-basis
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Fig. 1. A 3D inhomogeneity \( V' \) embedded in a layered-earth model. The structure is energized by a loop or grounded-wire source and components of the magnetic and/or electric field are received at various locations. The conductivity of the model is a sum of its anomalous conductivity \( \sigma_a \) and the background conductivity \( \sigma_b \).

representation and the method of moments. Displacement currents are assumed to be negligible and permeability is assumed to be \( 4\pi \times 10^{-7} \text{ H/m} \). The electric field at the centre of each of \( N \) prismatic cells composing the inhomogeneity is given by the solution of the following system of equations:

\[
E(r_n) = E'(r_n) + \sum_{a=1}^{N} \sigma_a \int_{V_n} G'(r_n, r') \, dv E(r'),
\]

(2)

where explicit dependence of the fields and Green's functions on frequency has been omitted for clarity, \( m = 1, \ldots, N \), and \( E(r_n) \) is the internal electrical field at the centre of the \( n \)th cell \( V_n \) over which it and \( \sigma_a \) are assumed constant.

In the second step of the forward solution, model responses consisting of 'external' electric and/or magnetic fields are computed using:

\[
F(r) = F'(r) + \sum_{a=1}^{N} \sigma_a \int_{V_n} G'(r, r') \, dv E(r_n),
\]

(3)

In this work the problem of estimating the geometrical characteristics and anomalous conductivity of the inhomogeneity is considered; the background conductivity structure is assumed to be known. Consequently all measured or computed responses are reduced to scattered (total less primary) values. Collecting together the (scattered) responses that pertain to a given model, one has

\[
F_i = \sum_{a=1}^{N} \sigma_a Z_{ij} \quad i = 1, \ldots, M
\]

(4)
where $M$ is the product of the number of frequencies, sources, and receivers used in conjunction with this model and

$$Z'_m = Z'(r_1, r_n) = \left\{ \sum_{n=1}^{N} G(r_1, r_n) \right\} E(r_n).$$

$F_i$ has $x$, $y$ and $z$ components, although only those components measured are of interest. Each component is a complex function of frequency, with both an in-phase and a quadrature part.

In the 3D electromagnetic inverse problem, we are given $M$ measured responses and must solve for the $N$ unknown anomalous conductivities.

**Non-linear Inversion**

Many geophysical inverse problems are non-linear in the sense that the response is not a linear function of the model parameters. In this case $Z'_m$ in (4) is a function of the internal electric field, which in turn is a function of $\sigma_a$. The approach used here to solve the 3D electromagnetic inverse problem posed above is one in which the parameters are iteratively refined using a sequence of least-squares approximations. Each approximation consists of minimizing the objective function

$$S^k = \sum (F_k - \tilde{F}_k)^T (\mathbf{D} \cdot \mathbf{D}) (F_k - \tilde{F}_k),$$

where $F_k$ is the difference at the $k$th iteration between the measured data ($F$) and the $k + 1$ model response ($\tilde{F}$). This response is approximated by a first-order Taylor series expansion in $\sigma_a$ about the $k$th model response

$$F^{k+1} = F^k + Z' \delta \sigma_a,$$

where

$$[Z']_{ij} = \frac{\partial F_i^k}{\partial \sigma_a^j}$$

is an element of the sensitivity matrix (a vector with $x$, $y$ and $z$ components) and

$$\delta \sigma_a = \sigma_a^{k+1} - \sigma_a^k$$

is the parameter change vector. Note that the scalar quantity $\sigma_a^j$ refers to the anomalous conductivity of the $j$th cell at the $k$th iteration, while the vector quantity $\sigma_a^k$ refers to the anomalous conductivities of all $N$ cells at the $k$th iteration.

$\beta$ in (5) is a constant and $D$ is a weighting matrix whose specifications depend on whether a penalized, damped, or smoothed least-squares solution is sought. Regardless of which solution is used, it is possible to show that minimizing (5) is equivalent to minimizing the quadratic function

$$S^k = \frac{1}{2} \delta \sigma_a^T (Z'^T Z' + \beta \mathbf{D}^T \mathbf{D}) \delta \sigma_a - \delta F^T Z' \delta \sigma_a,$$

where

$$\delta F = F_k - \tilde{F}_k.$$
This expression is amenable for solution using the algorithm of Fletcher and Jackson (1974) based on quadratic programming. In their scheme the quadratic is minimized in the least-squares sense subject to upper and lower bounds on the variables.

When using the integral-equation formulation in conjunction with the electromagnetic inverse problem, it is tempting to try to solve the system of equations in (4) directly for the unknown conductivities using an assumed internal electric field, and with the model responses replaced by the measured responses. This approach for 3D inversion may be characterized by poor convergence. This is unfortunate because such a scheme does not require partial derivatives which are usually difficult to obtain in a cost-effective manner.

INTEGRAL-EQUATION SOLUTION FOR THE INVERSE PROBLEM

Some progress in solving inverse problems similar to the one discussed in this paper using integral-equations has been reported by Barthes and Vasseur (1978) and Johnson et al. (1983) who estimate an initial model, estimate the internal electric fields, and then solve a system of equations like (4) for the physical property characterizing the inhomogeneity. This yields a new model for which the internal fields need to be estimated and thus begins a new iteration. Non-linearity is accommodated by the iterative nature of the procedure. In some situations, this technique yields a useful inverse solution (see Acknowledgements). However, in other cases the technique is characterized by poor convergence, particularly for high-contrast models.

Consider the kth iteration and a matrix formulation of (4):

$$Z \sigma_k^i = F^k,$$

where $Z$ is an $M$ by $N$ matrix whose elements are the vectors $Z^i$. As noted in the preceding discussion, one might be tempted to proceed by solving

$$Z \sigma_k^{i+1} = F_d$$

in, for example, the least-squares sense for new conductivity estimates.

The problem with this development is that (9) is not necessarily a relevant expression in the context of a non-linear inverse problem. At each iteration of the non-linear inverse solution described earlier one must compute the elements of the sensitivity matrix of partial derivatives. Differentiating (4) according to (6),

$$Z_{ij}^{k} = \frac{\partial F_{i}}{\partial \sigma_{j}} = Z_{ij}^{k} + \sum_{n=1}^{N} \sigma_{m} \frac{\partial Z_{in}^{k}}{\partial \sigma_{j}}.$$  

(10)

It is evident that the sensitivity matrix is the sum of $Z$ and a second matrix whose elements are weighted by the anomalous conductivities. This term arises because the internal electric field is a function of the anomalous conductivities, which necessitates the non-linear approach that is used here. Equation (10) reveals why the simpler (quasi-linear) approach described above, which does not require partial derivatives, may solve the non-linear inverse problem in some cases. If the
contrast is low, i.e. if \( \sigma_a \) is small, then \( Z' \) might be approximately equal to \( Z \). Alternately, if the primary electric field in the earth well approximates the internal electric field, then the variation of the internal field with respect to a perturbation in the anomalous conductivity of the model would likely be quite small. This variation relates to the magnitude of the \( \partial Z' \) terms in (10) and under these conditions \( Z' \) might again be approximately equal to \( Z \). If such conditions are not satisfied, the error in neglecting the second term in (10) manifests in an error in the gradient of the objective function being minimized in the inverse solution. Therefore the direction and/or step length used to advance the quasi-linear solution toward a minimum will be in error and convergence will be slow, at best.

To circumvent these problems here the sensitivity matrix is computed in full, using both terms in (10). The first term requires a forward solution for the current set of parameter estimates, i.e. a matrix solution for the internal electric fields (2) followed by a direct calculation of the \( Z'_a \) (4). Solving for the second term in (10) requires that partial derivatives of the internal electric field with respect to each of the anomalous cell conductivities be computed. From (2),

\[
\frac{\partial E(r_m)}{\partial \sigma_a} = \left\{ \left. \int_{V_j} G'(r_m, r') \, dv' \right\} E(r_j) + \sum_{n=1}^{N} \sigma_n \left\{ \left. \int_{V_n} G'(r_m, r') \, dv' \right\} \frac{\partial E(r_n)}{\partial \sigma_a} \right. \tag{11}
\]

The virtue of this expression is that it is identical to the system of equations solved for the internal electric field (2) with the field replaced with its partial derivative, and the primary field replaced with

\[
\left\{ \int_{V_j} G'(r_m, r') \, dv' \right\} E(r_j),
\]

which is available once the solution for the internal electric field has been computed. Linear combinations of the electric-field derivatives compose the \( \partial Z'_a \) terms in (10). For example, the x-component of the partial derivative with respect to the \( j \)th parameter is

\[
\frac{\partial (Z_a)}{\partial \sigma_j} = \frac{\partial E_x(r_n)}{\partial \sigma_j} \Gamma'_{x,n} + \frac{\partial E_y(r_n)}{\partial \sigma_j} \Gamma'_{y,n} + \frac{\partial E_z(r_n)}{\partial \sigma_j} \Gamma'_{z,n}, \tag{12}
\]

where, for example,

\[
\Gamma'_{x,n} = \int_{V_n} G'_{x,n}(r_1, r') \, dv'.
\]

Thus, to compute both the model response and the sensitivity elements at each iteration, a single forward (matrix) solution is computed followed by \( N \) additional back substitutions. This process is repeated for all frequencies and sources used. Such a procedure is more efficient and accurate than one based on difference approximations for the partial derivatives. In fact, the derivatives are as accurate as the fields. Similar approaches for the 2D electromagnetic inverse problem based on finite-difference and finite-element formulations are described by Jupp and Vozoff (1977) and Oristaglio and Worthington (1980), respectively.
THE INVERSE PROCEDURE

The first step in interpreting the conductivity and shape of a 3D inhomogeneity embedded in a layered earth from a set of magnetic-field and/or electric-field measurements is to obtain an initial model. From a set of transient soundings, the interpretation technique described in Eaton and Hohmann (1989) is used to obtain an approximate image of the distribution of resistivities in the earth beneath the survey. I will refer to this result as the image solution. It is similar to the result obtained using the techniques described by Macnae and Lamontagne (1987) and Nekut (1987), and works best on models that are quasi-layered. At positions where the source is poorly coupled with the inhomogeneity it is possible to obtain an estimate of the background conductivity structure. Where the model strongly deviates from a quasi-layered structure, extraneous features in the estimated resistivity section, such as a resistive overshoot beneath a conductive body, can occur. The result is an initial model that is usually much larger than the actual inhomogeneity.

The second step in the inverse procedure is optional. It consists of reducing the size of the anomalous region outlined by the image solution, by eliminating some of the cells that have no anomalous conductivity in the true model, i.e. those that are completely contained within the layered regions of this model. This is done by choosing the weighting matrix $D$ in (5) to be constructed from the eigenvectors of $Z^T Z$ associated with the $N - L$ smallest singular values of $Z$. This has the effect of penalizing those solutions with features that produce negligible effect on the calculated response, and was utilized by Fisher and Howard (1980) for inverting gravity data. Because only the 3D inhomogeneity in the true model contains scattered current and because the true model is the one that produced the data being fit, models that disperse the scattered current over an anomalously large volume are likely to be less geoelectrically feasible than those that do not. The result is a 'geometrical' solution in which the conductivities of the cells that are not eliminated remain fixed as the size and shape of the model changes with successive iterations.

Because we are not concerned at this stage with refining the conductivity estimates by fitting measured and model responses, but rather with characteristics of the penalized solution, it is acceptable to utilize (g) rather than the more computationally-intensive non-linear solution. Accordingly, at the $k$th iteration the quadratic

$$ S^k = \frac{1}{2} (\sigma_{a+1}^k)^T (Z^T Z + \beta D^T D) \sigma_{a+1}^k - F^T \sigma_{a+1}^k $$

is minimized rather than (7). The effect of the penalty term is to push the solution $\sigma_{a+1}^k$ toward the null vector. The smaller the cutoff index $L$, the larger the penalty constant $\beta$ (proportional to the square of the $L$th singular value), and the greater the push. The cutoff index $L$ is chosen to be relatively small at the first iteration and in such a way that the ratio of the largest (first) singular value to the $L$th singular value remains approximately the same in successive iterations.

At each iteration of geometry inversion there is a tendency for the anomalous conductivity estimates for some of the cells that are extraneous in the model to be set to zero, while the estimates for the essential cells are not. This provides one
criterion for eliminating a few extraneous cells. There is also an association of some of the extraneous cells with relatively large values of the gradient of

$$(Z^T Z + \beta (D^T D) \sigma^2_k + 1 - Z^T F_d).$$

This quantity is readily available from Fletcher and Jackson's (1974) algorithm and provides an additional criterion for eliminating some of the cells.

I refer to the third step of the inverse procedure, in which estimates of the conductivity of the remaining cells are refined, as conductivity inversion. At each iteration, new conductivity estimates are obtained by minimizing (7). By choosing $D = I$, the damped least-squares solution is realized. This particular solution is not necessarily optimal and a smooth least-squares solution may also be formulated for this problem (Eaton 1987). The geometry of the model, i.e. the distribution and size of the cells, remains fixed in the final step of the inverse procedure.

For stability it is desirable to be able to control how much the resistivities can vary from one iteration to the next (Glenn et al. 1973). The method used here is to define bounds on the parameters so that each cell's resistivity at a given iteration can only differ from the previous iteration's value by, at most, a factor of $\sqrt{10}$. Computational aspects of the least-squares solution are improved by scaling the parameters so that unity appears on the diagonal of $(Z^T Z)$ in (7).

When the 3D inhomogeneity possesses two orthogonal, vertical planes of symmetry it is possible to significantly reduce the size of the inverse problem. Using group theory (Tripp and Hohmann 1984), the number of calculations and size of memory needed to solve for the model response and sensitivity matrix at each iteration is reduced. Furthermore, the number of parameters to be estimated in the inversion is reduced by a factor of 4. This symmetry constraint applies to the geometry and conductivity of the anomalous region, but does not apply to the distribution of sources and receivers used.

**Results**

The inverse procedure described in the previous section is demonstrated here using synthetic data generated for 3D models using the computer modelling program described by Newman, Hohmann and Anderson (1986).

Model inadequacy, i.e. using a model that can not accurately simulate the physical processes that contribute to a measured response, is a severe problem for parameter estimation techniques such as this one. For example, if the boundaries of the initial model agree with the true model, the cells must be small enough to yield accurate results, based on using pulse-basis functions in the forward and inverse solutions, and small enough to track the spatial variation of conductivity in the model. If the boundaries of the initial model do not coincide with the true model, problems arise because the precise solution for the resistivity of those cells straddling the boundaries is ambiguous and because portions of the true model may not even be represented by the initial model. Using a large number of parameters reduces the likelihood of these problems, but may cause instability in the least-squares inversions and may require exorbitant computing resources. As indicated by
Oristaglio and Worthington (1980), the inadequate model yields a damped least-squares solution that for a buried conductor may contain extreme resistivity values, i.e., the body cells are too conductive and the cells surrounding the body are too resistive. In such a situation a smooth least-squares solution, similar to that described by Constable, Parker and Constable (1987), may yield a more geoelectrically satisfying result.

It is important to realize that the problem of model inadequacy is minimized by using a 3D scheme to interpret what are often, in practice, 3D responses. Using 1D or 2D approximations would worsen matters in this regard. By using cells that are sufficiently small and strategically positioned, the effects of model inadequacy on the following results are negligible.

Model 1

In this example several different tests of the inverse procedure are discussed. Results are presented from each step of the procedure.

The 3D structure illustrated in Fig. 2 consists of a topographical depression in moderately resistive (250 Ωm) terrain, filled with more conductive (2.5 and 50 Ωm) material. The basin is equidimensional in plan view. The objective of a hypothetical electromagnetic survey would be to map the shape of the basin and to determine the extent of the 2.5 Ωm fill.

The image solution is superimposed on Fig. 2 and was generated from a profile of 750 m central-loop soundings over the basin. The contours clearly exhibit a 3D distortion from otherwise 1D resistivity variations. The conductivity structure of the host (a 400 m thick, 50 Ωm layer overlying the 250 Ωm basement) is poorly estimated from a depth of 400 m to approximately 4000 m. This is a common characteristic of the image solution associated with the transition from a conductive to a resistive layer with increasing depth. The image solution was computed in only a few minutes using a VAX 11/785 computer.

Symmetry was utilized and a 3D region extending from \( x, y = 0-1800 \) m and \( z = 250-7500 \) m was discretized. Resistivities were assigned to the cells from the image solution. The background structure was interpreted to consist of eight layers with resistivities varying smoothly from 50 to 250 Ωm, as noted in Fig. 2. Figures 3 and 4 illustrate portions of the initial model and the geometrical solution based on a data set generated using nine 750 m loops at \( x, y = 0 \) m, 1000 m and 2000 m, and three-component, central-loop magnetic-field measurements at 0.1, 1.0, 10 and 100 Hz. In the second step of the inverse procedure the number of cells in the model (85 per quadrant) is held fixed so that when a cell is eliminated, another cell elsewhere in the model is divided into two cells. In this way the cells become smaller with each iteration of the geometrical inversion and the forward solution becomes more capable of yielding accurate model responses. The improvement in Fig. 4 is considerable, having only a few extraneous cells directly above, directly below, and immediately adjacent to the basin. The extraneous resistive feature beneath the basin, produced by the image technique, has been eliminated by geometry inversion. This is the result of a single iteration using the penalized least-squares algorithm and a cutoff index of 10. No additional cells were eliminated on the second iteration.
Fig. 2. Model 1. Estimated resistivities (image solution) superimposed on a cross-section through the true model. An estimate of the background conductivity structure, based on the image solution, appears on the left-hand side of the section. The depth scale is linear from the surface down to a depth of 100 m, and logarithmic thereafter.
FIG. 3. Model I. Horizontal and vertical sections taken through one quadrant of the initial model for geometry inversion, constructed from the image solution shown in Fig. 2. The value at the centre of each cell specifies its resistivity (Ωm).

The geometrical solution was computed in approximately one hour using the VAX computer. By utilizing symmetry to define the model and a symmetrical distribution of source and receivers, the inverse solution is constrained to be symmetrical as well.

Convergence of a damped least-squares solution is generally acknowledged when the size of the error vector (e in (5)) becomes small relative to the noise level in
FIG. 4. Model I. Horizontal and vertical sections taken through one quadrant of the model representing the geometrical solution. Compare with Fig. 3.

the data. The mean-square-error (MSE) residual is one measure of the size of this misfit between the measured and computed model responses:

$$\text{MSE} = \frac{1}{M} \sum_{i=1}^{M} e_i^2 = \frac{1}{M} \sum_{i=1}^{M} \left( \delta F_i - \sum_{j=1}^{N} Z_{ij} \delta x_{ij} \right)^2,$$

(14)

where $M$ is the total number of measurements and $N$ is the total number of parameters, i.e., cells in the model. A summation over each component $(x, y, z)$ used, and
over both the in-phase and quadrature parts of the response is implicit in (14). Convergence is also indicated when all of the parameters are 'free', i.e. when all of the parameter changes are less than the maximum allowed as described in the previous section.

The model shown in Fig. 4, with some of the cells combined to reduce the total number of cells to a more manageable level (53 per quadrant), was used as the initial model for refining the conductivity estimates. This revised initial model is presented in Fig. 5. Using only two of the nine sources ($x = 0, 1000 \text{ m}$) and two

![Diagram](image_url)

**Fig. 5.** Model 1. A refinement of the model illustrated in Fig. 4, used as an initial guess for conductivity inversion.
frequencies (1 and 30 Hz) to reduce computing costs, a few iterations using the eight-layer background model were computed. The resistivity estimates for the inhomogeneous region were much too large in the conductivity solution (not shown). This was because the interpreted 1D host is too conductive from a depth of 400 m to 4000 m. Making the background structure too conductive results in a bias in the scattered model response computed at each iteration. The effect of this bias is to make the 3D region in the conductivity solution appear to be much more resistive than it actually is. In order to refine the estimates of the cell conductivities, the background conductivity structure must be accurately known.

With the correct background conductivity structure, i.e., a two-layer host, geometry and conductivity inversions were repeated. The geometrical solution (not shown) is a slight improvement on the result shown in Fig. 4, having fewer extraneous cells above and adjacent to the basin. However, the difference in the geometrical solutions arising from this sort of error in the conductivity of the host is not significant.

Starting with the initial model of Fig. 5, the damped least-squares algorithm (conductivity inversion) yielded the resistivity estimates presented in Fig. 6 after 17 iterations. Each iteration was computed in about 30 minutes using the VAX computer. The resistivity of the upper portion of the basin is very well resolved; the variances of the (scaled) parameters associated with the top layer of cells (z = 250–400 m) are of the order of $10^{-18}$ with corresponding values of the Dirichlet Spread Function or DSF (Menke 1984) ranging from $10^{-5}$ to $10^{-4}$. Small values of these quantities correspond to low uncertainty and good resolution of the resistivity estimates. As expected, with depth these measures of uncertainty and resolution become progressively worse. The second tier of cells (z = 400–700 m) have associated variances of about $10^{-18}$ and values of the DSF ranging from $10^{-4}$ to $10^{-2}$. The nine cells representing conductive (2.5 Ωm) fill have variances ranging from $10^{-16}$ to $10^{-15}$, while values of the DSF vary widely from $10^{-2}$ to 40. For the larger cells representing the resistive walls of the basin, these statistical indicators improve and the resistivities attain their true values. Beneath the basin all of the resistivity estimates are poorly resolved. In spite of this, reconstruction of the model is good. The interpretation in Fig. 6 compares favourably with the true model in Fig. 2.

Reducing the amount of data (from $M = 216$ to 54) to consist of a single frequency (3 Hz) and source ($x$ and $y = 0$ m) with the same nine receivers used in the previous example yielded less satisfactory results (not shown). In this test the data set contained insufficient information to resolve most of the 3D structure. The MSE residual associated with the conductivity solution was about 10 times smaller than for the previous data set, but the resistivity estimates were much less accurate. The number of free parameters was also smaller: 30 rather than 52. Many of the deep cells had large resistivities, ranging from 1000 to 100,000 Ωm, although the resistivity of a few of the cells coinciding with the conductive fill was of the order of 1–3 Ωm. The average resistivity of the “free” cells higher in the section, i.e., those at depths of 250–700 m, was 58.3 Ωm with a standard deviation of 31.3 Ωm. The estimates for these shallow cells would likely have been better if a higher frequency had been selected.
Adding a small amount of random noise to the sounding data has little effect on the resistivity estimates obtained using image inversion, particularly if multiple receivers are used for each source position. 10% and 25% uniformly-distributed random noise was added to the multifrequency data sets used in the geometry and conductivity inversions. The effect of random noise on refining the size and shape of the inhomogeneity was insignificant, i.e. identical cell eliminations occurred in these tests. The conductivity solutions were acceptable in the sense that the average resistivity of different portions of the 3D structure was approximately correct. For
example, in adding 25% noise, the average resistivity of the cells representing the conductive (2.5 Ωm) fill was 3.7 Ωm with a standard deviation of 31.9 Ωm after five iterations. With no noise, the average and standard deviation of these same estimates after five iterations were 3.3 and 11.0 Ωm, respectively.
**Model 2**

This model consists of a 1 Ωm vertical dike with a strike extent of 320 m, a depth extent of 100 m, and a width of 40 m. The dike is buried 80 m deep in a 100 Ωm half-space. To demonstrate several additional features of the final step of the inverse procedure (conductivity inversion), the initial model superimposed on the true model in Fig. 7 is used. This model has about the same time constant as the true model but is too thick, too resistive, and has too great a depth extent. In this example it is assumed that the anomalous conductivity does not vary along strike and that the strike length of the inhomogeneity is known. The cells that compose the initial model are organized into rectangular cylinders oriented parallel to the strike of the structure. The anomalous conductivities of these cylinders are the parameters in the inversion, rather than the conductivities of the individual cells. The quantities associated with a cylinder as determined at each iteration, such as the parameter change and gradient, are inherited by each of its cells. Applying symmetry as well leaves 12 unknown anomalous conductivities, rather than 192 (4 cells per cylinder × 12 cylinders per quadrant × 4 quadrants), for which to invert.

In addition to illustrating how the size of the inverse problem may be reduced, a grounded-source interpretation based on three-component magnetic-field measurements and horizontal electric-field measurements confined to a centred profile is described. The source-receiver configuration is illustrated in Fig. 7 and data at a single frequency (100 Hz) were computed from the true model.

The results from several iterations of the conductivity solution, using the damped least-squares algorithm and the correct host resistivity, are presented in Fig. 8. In this figure cross-sections of the model with resistivity estimates for each cylinder are shown. The number of 'free' parameters and the MSE residual at each of these iterations are noted in the figure captions. Observe that initially there are no free parameters. As the inversion proceeds, the residual decreases and the number of free parameters increases. Resistivity variations, from 4 to 13 Ωm or from 13 to 40 Ωm for example, reflect the constraint on the maximum allowable parameter changes described earlier. It is important to note that the process of convergence to the true model does not necessarily consist of a monotonic variation of the resistivities from the initial values to the true values. This is illustrated, for example, at iteration 5, where the resistivity of a cylinder residing within the dike has decreased from 4 to 0.21 Ωm, which is less than the true resistivity (1 Ωm). In the same number of iterations the resistivity of the adjacent cylinder, residing outside the dike, has increased from 4 to 12.65 Ωm, which is greater than the background resistivity (100 Ωm). Although this divergent behaviour often occurs in the process of refining the resistivity estimates, it does not prevent convergence to the true model. On the contrary, it seems to improve convergence. In this example all resistivity estimates are correct to within 1% at the 15th iteration. When the resistivities at all iterations are constrained to lie between 0.5 and 500 Ωm, which clearly includes both the initial and the true model values, the damped least-squares algorithm abruptly converges at the sixth iteration. In this solution (not shown) the dike cells are generally too conductive and the external cells are generally too resistive. In fact, several of the parameters are constrained against the lower or upper bound.
(a) Iteration = 0 (initial model).

(b) Iteration = 1, NF = 0, MSE = $7.5 \times 10^{-11}$.

(c) Iteration = 2, NF = 1, MSE = $3.4 \times 10^{-11}$.

(d) Iteration = 5, NF = 8, MSE = $6.5 \times 10^{-15}$.

(e) Iteration = 8, NF = 11, MSE = $5.2 \times 10^{-18}$.

(f) Iteration = 11, NF = 12, MSE = $3.0 \times 10^{-19}$.

Fig. 8. Model 2. Selected conductivity inversion results (model cross-sections) illustrating convergence of the damped least-squares solution. The value at the centre of each cylinder specifies its resistivity ($\Omega$m). At a given iteration, NF specifies the number of 'free' parameters while MSE specifies the mean-square-error residual.
CONCLUSIONS

The inverse procedure described in this paper for controlled-source electromagnetic data can be used to interpret the geometry and conductivity of a 3D structure embedded in a layered earth, without resorting to strictly layered-earth or free-space models.

Rarely can a 3D geoelectrical model that unequivocally reflects the distribution of resistivities in the earth be determined in practice. However, it is often useful to be able to establish the approximate resistivity and geometry of 3D structures in the earth. Image and geometry inversions usually do a good job in this regard using reasonable computing resources, in spite of noise in the data or errors in the assumed conductivity structure of the host.

The usefulness of conductivity inversion depends on the adequacy of the model used to produce a fit to the observations. The problem of model inadequacy is difficult to combat with the scheme presented here without having to use a large number of cells and consequently large computing resources. In experiments based on synthetic data there does not appear to be much of a problem with random noise in the measurements or with non-uniqueness, i.e. with converging to a model that is dissimilar to the true model but that produces as good a fit to the measurements, provided a sufficient data set is available. Systematic noise (bias) in the responses due to, e.g. an error in the assumed background model, can influence the conductivity solution in a deleterious fashion. The cost of doing this type of 3D inversion must be justified in terms of the quality of the data set to be interpreted, and in terms of the improvement in electromagnetic interpretation that such a scheme potentially offers.

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An anonymous reviewer recognized that the approximate technique for solving the electromagnetic inverse problem, i.e. solving (9), does work for some types of models and he provided an enlightening example for 2D (E-polarization) inversion.

REFERENCES


