MIGRATION BY ANALYTIC CONTINUATION THROUGH A VARIABLE BACKGROUND MEDIUM

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ABSTRACT

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Conventional migration of wavefields is based on depth extrapolation of the upgoing field in reverse time. This extrapolation provides us with a means to determine the positions of reflectors and diffraction points and, therefore, to produce an image of a geological cross-section. However, conventional depth extrapolation allows us to restore only some transformation of the field in the subsurface rather than the true field. For example, this approach rules out the proper imaging of multiple reflections. Meanwhile, there is reason to expect that a different reconstruction of the seismic wavefield - reconstruction by analytic continuation can yield a more comprehensive image of a medium. An exact depth extrapolation based on analytic continuation could contribute to the restoration of the true process of seismic wave propagation in a medium. In this case, multiples do not 'pass' the layers where they have been formed and, hence, they cannot generate any fictitious reflecting boundaries. Here, we describe a method for doing migration of seismic wavefields in the frequency domain by analytic continuation through a medium with vertically variable velocity.

KEY WORDS: migration, analytic continuation, multiples elimination, boundary problem.

INTRODUCTION

The conventional procedure for the migration of wavefields is usually based on the depth extrapolation of the upgoing field in reverse time (Claerbout, 1985). However, this extrapolation allows us to restore only some transformation of the field in the subsurface rather than the true field. Also,

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one-way field extrapolation cannot reconstruct correct amplitudes of the seismic waves (Larner et al., 1981; Kozloff and Baysal, 1982). In some situations (e.g., when we wish to image multiples) it can be useful to restore the true distribution of the total seismic wavefield in the subsurface. This restoration can be accomplished by the method of analytic continuation. In analytic continuation, the field distribution in the lower halfspace with given background velocity is restored from the known values of the field on the earth's surface. The method is related to the analytical theory of the functions of real or complex variables, but in this paper we discuss only the straightforward implementation needed for the solution of the migration problem.

Some aspects of this problem have been discussed in earlier papers published by Kozloff and Baysal (1982) and by Zhdanov and Matusevich (1984). A comprehensive analysis of this problem has been developed by Wapenaar and Berkhout (1985, 1986) and by Wapenaar et al. (1987). A more detailed introduction to this subject can be found in Zhdanov (1988); and Zhdanov et al. (1988). Here, we focus on the explicit method of migration by analytic continuation in a layered medium with a vertically variable velocity, based on the Green's function decomposition. This new technique makes it possible to construct an analytical solution for the problem of two-way wave extrapolation in a medium with a specific depth-dependent velocity.

FORMULATION OF THE PROBLEM

The problem of analytic continuation can be formulated as a boundary-value problem for the wavefield.

Suppose the seismic field (vertical component of displacement or pressure in the case of an acoustic model) and its normal derivative are given on the surface of the layered half-space with a given vertical profile of velocity variations (note that the practical ways of determining the boundary values u^1 will be discussed later):

$$u(x,y,0,\omega) = u^0(x,y,\omega) \quad , \quad (\partial/\partial z)u(x,y,z,\omega)\big|_{z=0} = u^1(x,y,\omega) \quad . \tag{1}$$

Reconstruction of the field $u(x,y,z,\omega)$ is required inside any layer in the earth. The solution of this problem can be divided into two stages:

a) continuation of the seismic field into a given layer with a specified depth-dependent velocity $V_\alpha(z)$ based on equation

$$\nabla^2 u(x,y,z,\omega) + \{\omega^2/V_\alpha^2(z)\}\, u(x,y,z,\omega) = 0 \quad , \quad L_{\alpha-1} \le z \le L_\alpha \quad , \tag{2}$$

where L_α means the depth to the bottom of the α layer.

b) recalculation of the field and its normal derivative across an interface by applying the proper boundary conditions on interfaces.

Therefore, the solution of the stated problem comes down to the field continuation within the limits of one given layer.

WAVEFIELD IN THE FREQUENCY SPACE TIME DOMAIN

We can present the field $u(x,y,z,\omega)$ in the form of the Fourier transform over space variables x,y

$$\tilde{u}(k_x,k_y,z,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x,y,z,\omega)e^{-i(k_x x+k_y y)}\,dxdy \quad . \tag{3}$$

In the space-frequency domain equation (2) can be written as

$$L\tilde{u}(k_x,k_y,z,\omega) = 0 \quad , \tag{4}$$

where L is the one-dimensional Helmholtz operator

$$L \equiv (\partial^2/\partial z^2) + (\omega^2/C^2(z)) \quad ,$$

and

$$(1/C^2(z)) = (1/V^2(z))\{1 - V^2(z)[(k_x/\omega)^2 + (k_y/\omega)^2]\} \quad .$$

Note that C is the function of vertical coordinate z , spatial frequencies k_x, k_y and temporal frequency ω, but for the purposes of solving the differential equation in z we write it only as C(z). Correspondingly, boundary conditions (1) will take the form

$$\tilde{u}(k_x,k_y,0,\omega) = \tilde{u}^0(k_x,k_y,\omega) \quad , \quad \tilde{u}'(k_x,k_y,0,\omega) = \tilde{u}^1(k_x,k_y,\omega) \quad , \tag{5}$$

where the prime denotes the vertical derivative. Note that the seismic wave source term is included in the boundary conditions at the surface.

Our goal is to determine $\tilde{u}$ everywhere inside the first layer, where the function C(z) is the continuous function of depth, from the given values of $\tilde{u}^0$ and $\tilde{u}^1$ on the earth's surface. To solve this problem we apply the 1-D Green's theorem and corresponding Green's function.

SOLUTION OF THE BOUNDARY VALUE PROBLEM

Let us apply Green's theorem in one dimension to the first layer (Bleistein, 1984):

$$\int_0^{h_1} [\tilde{u}(z)L\tilde{g}(z,\zeta) - \tilde{g}(z,\zeta)L\tilde{u}(z)]dz = \tilde{u}(z)\tilde{g}'(z,\zeta)\Big|_0^{h_1} - \tilde{g}(z,\zeta)\tilde{u}'(z)\Big|_0^{h_1} . \quad (6)$$

Here, $\tilde{u}(z) = \tilde{u}(k_x,k_y,z,\omega)$ and $\tilde{g}(z,\zeta) = g(k_x,k_y,z,\zeta,\omega)$ is the Green's function that satisfies the equation

$$L\tilde{g}(k_x,k_y,z,\zeta,\omega) = -\delta(z - \zeta) \quad . \quad (7)$$

Substituting equations (4) and (7) into (6) gives

$$\tilde{u}(\zeta) = \tilde{g}(z,\zeta)\tilde{u}'(z)\Big|_0^{h_1} - \tilde{u}(z)\tilde{g}'(z,\zeta)\Big|_0^{h_1} . \quad (8)$$

We now have two key issues. The first is how to determine the Green's function. This problem can be solved using the high-frequency asymptotic (WKBJ) approximation for the Green's function (Bleistein, 1984). The corresponding expressions for the Green's function are given in the Appendix.

The second key issue is how to determine the values of the field $\tilde{u}$ on the bottom of the layer $z = h_1$ in equation (8). Evidently, we can solve the boundary value problem (the recalculation of the field $\tilde{u}$ inside the earth), if we know the field's values on the bottom of layer $z = h_1$. But we only have the recorded data u on the earth's surface $z = 0$. To overcome this difficulty we apply a special method of transformation of the portion of the right-hand side in (8), pertaining to the bottom of the layer, so that it is expressed in terms of values at the surface. As is shown in the Appendix, we can express the field values from level $z = h_1$ in terms of the values at the earth's surface by decomposing the Green's function into the multiplication of two functions that depend on z and ζ separately, and by using the Green's theorem (6) once again. The Appendix gives a detailed mathematical explanation of this transformation.

Following this procedure, we have from equation (8) the following expressions for the analytic continuation of the wavefield into the first layer:

$$\tilde{u}(\zeta) = \tilde{u}^0 \sqrt{\{C(\zeta)/C(0)\}} \cos[\omega\phi(\zeta)] + \tilde{u}^1[\sqrt{\{C(\zeta)C(0)\}} / \omega] \sin[\omega\phi(\zeta)] \quad , \quad (9)$$

where $\phi(\zeta) = \int_0^{\zeta} d\xi/C(\xi)$.

In the same way, we can obtain the formula for the vertical derivative of the field inside the first layer. Similar expressions can be applied to the wavefield continuation from the top of the α layer to a position within this layer:

$$\tilde{u}(\zeta) = \tilde{u}(h_{\alpha-1}) \sqrt{\{C_\alpha(\zeta)/C_\alpha(h_{\alpha-1})\}} \cos[\omega\phi_\alpha(\zeta)] +$$
$$\tilde{u}'(h_{\alpha-1})[\sqrt{\{C_\alpha(\zeta)C_\alpha(h_{\alpha-1})\}} / \omega] \sin[\omega\phi_\alpha(\zeta)] \quad , \quad (10)$$

where $$\phi_\alpha(\zeta) = \int_{h_{\alpha-1}}^{\zeta} d\xi/C(\xi) \quad . \tag{11}$$

Thus, using formulae (9) through (11) for continuation inside layers, along with proper boundary conditions on the interfaces, we can analytically continue the wavefield u from the surface of the earth through a variable background medium to any internal point of the earth's interior (as long as $C^2(z) > 0$, see the Appendix).

For analytic continuation of recorded seismic data, we apply to equations (9) and (10) the inverse Fourier transform from the wavenumber-frequency domain (k_x,k_y,ω) to the space-time domain (x,y,t).

BOUNDARY CONDITIONS

In practice, only one of the functions u^0 or u^1 is recorded on the earth's surface. Computation of the remaining field requires an additional geophysical assumption.

Here, we consider two possible models for the boundary conditions. One is based on the so-called *free surface conditions* for the vertical component of displacement wherein the vertical derivative of the wavefield on the surface of the solid earth is equal to zero

$$u^1(x,y,\omega) = 0 \quad . \tag{12}$$

Relation (9) is then

$$\tilde{u}(\zeta) = \tilde{u}^0\sqrt{\{C(\zeta)/C(0)\}}\,\cos[\omega\phi(\zeta)] \quad . \tag{13}$$

Another model has been discussed by Kozloff and Baysal (1983). They suggest that the velocity $V(z) = V(0) = const$ in some vicinity of the surface and that the recorded wavefield consists of upgoing waves only. For constant velocity, the general solution of equation (4) in the vicinity of $z = 0$ takes the form:

$$\tilde{u}(k_x,k_y,\zeta,\omega) = \mathbf{u}^+(k_x,k_y,\omega)\exp[i\omega\{\zeta/C(0)\}] + u^-(k_x,k_y,\omega)\exp[-i\omega\{\zeta/C(0)\}] \quad . \tag{14}$$

Because we consider only the upgoing wavefield at the surface, we delete the first term in this equation, giving

$$\tilde{u}(k_x,k_y,\zeta,\omega) = u^-(k_x,k_y,\omega)\exp[-i\omega\{\zeta/C(0)\}] \quad . \tag{15}$$

Now we can calculate the vertical derivative of the wavefield at the earth's surface

$$\tilde{u}^1(k_x,k_y,\omega) = [-i\omega\{\zeta/C(0)\}]\ \tilde{u}^0(k_x,k_y,\omega) \quad . \tag{16}$$

Substituting (16) into (9), we have for the first layer,

$$\tilde{u}(\zeta) = \tilde{u}^0\sqrt{\{\{C(\zeta)/C(0)\}}\ \exp[-i\omega\phi(\zeta)] \quad . \tag{17}$$

Note that the formula (17) presents exactly Gazdag's phase-shift method for depth extrapolation of a seismic wavefield in the wavenumber-frequency domain. The main difference in this formula and the general formula for analytic continuation (8) is that the latter takes into consideration both upgoing and downgoing waves and, therefore, accounts for the multiple reflections that are produced by strong velocity contrasts. This is achieved by calculating the wavefield in equation (8) along the two surfaces: $z = 0$ and $z = h_1$ (but we should remember that, as a result of mathematical transformation, the integration over the plane $z = h_1$ is reduced to the integration over the surface of the earth $z = 0$). Conventional migration ignores the multiples, thus possibly resulting in the reconstruction of the false reflectors in the migration image (we will discuss this question in detail later).

The initial step of analytic continuation in the model suggested by Kosloff and Baysal (1983) is the same as phase-shift depth extrapolation. As soon as we cross the first strong velocity boundary, however, we have to apply boundary conditions and then extrapolate the field with continuation formula (9). From the physical point of view, this means that we take into consideration the multiples generated at the first strong velocity boundary, and at subsequent ones, as well.

Thus, we can combine conventional methods of depth extrapolation (if we wish to ignore the multiples in some depth interval) with analytic continuation through the structures with strong velocity contrasts. In this way, we aid the efficiency of analytic continuation. For media with continuously changing velocity $V(z)$ (i.e., $h_1 \to \infty$), analytic continuation with boundary condition (16) is identical to Gazdag's depth extrapolation (17).

MIGRATION OF TIME SECTIONS BY ANALYTIC CONTINUATION

It is important to recognize that in the general case analytic continuation is an ill-posed problem; that is, errors in the initial data can increase significantly during the continuation process. This follows from the behavior of evanescent waves, which are given by the exponentially varying solutions of the wave equation. The evanescent waves are defined by the condition

$$k_x^2 + k_y^2 > \omega^2/V^2(z) \quad . \tag{18}$$

The simplest way to overcome this difficulty is to eliminate the evanescent energy when implementing the analytic continuation algorithm (Kozloff and Baysal, 1982). Correspondingly, in this paper we use the high-frequency asymptotic approximation for the Green's function, which is valid only when

$$C^2(z) > 0 \tag{19}$$

or

$$k_x^2 + k_y^2 < \omega^2/V^2(z) \quad . \tag{20}$$

Thus, as with conventional depth extrapolation, by using the analytic continuation algorithm we eliminate the evanescent waves. The subsurface image then can be obtained based on a space-time structure analysis of the earth's total seismic wavefield. Consider the situation of a reflector on which a reflected wave is produced at the moment of direct wave arrival. Interference of the waves results in an increase (or decrease) of the amplitude of the total field at the reflector points in comparison with that of the surrounding points. This fact makes possible the use of analytic continuation for imaging of the earth's interior. We can restore the wavefield at each point of the section at the moment of direct-wave arrival. Then the locations of anomalous amplitudes of the field will show the location of the reflectors and the diffraction objects. We call this procedure *migration by analytic continuation*. This method of obtaining a migrated section (subsurface image $I_a(x,y,z)$) from time sections can be expressed mathematically by the formula:

$$I_a(x,y,z) = \int_{-\infty}^{+\infty} \delta[t - \tau(x,y,z)]u(x,y,z,t)dt \quad . \tag{21}$$

where $\tau(x,y,z)$ is the time of arrival of the direct wave at the point (x,y,z), $u(x,y,z,t)$ is the analytically continued seismic wavefield, and $\delta(t)$ is the Dirac delta function.

Thus, applying the inverse Fourier transform to equation (10) and then substituting the result into (21) we get

$$I_a(x,y,z) = (1/8\pi^3) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [\tilde{u}(h_{\alpha-1})\sqrt{\{C_\alpha(\zeta)/C_\alpha(h_{\alpha-1})\}} \cos\{\omega\phi_\alpha(\zeta)\} +$$

$$\tilde{u}'(h_{\alpha-1})\sqrt{\{C_\alpha(\zeta)C_\alpha(h_{\alpha-1})/\omega\}} \sin\{\omega\phi_\alpha(\zeta)\}]e^{i[k_x x + k_y y - \omega\tau(x,y,z)]} dk_x dk_y d\omega \quad , \tag{22}$$

which describes imaging based on the analytic continuation migration.

ELIMINATION OF MULTIPLE REFLECTIONS

The problem of multiple elimination has been extensively studied in a number of publications (see, for example, Anstey and Newman, 1967; Kennett, 1979; Verschuur et al., 1989; Wapenaar and Berkhout, 1985, 1986; Wapenaar et al., 1987; Verschuur, 1992). Migration by analytic continuation also provides an opportunity to eliminate the multiples that are produced by the (perfectly) reflecting free surface, as well as by the strong velocity contrasts in the lower parts of the section. This problem was discussed, for example, in Zhdanov and Matusevich (1984). Following Zhdanov et al. (1988), consider the two-layered model shown in Fig. 1. A line source is located in the lower layer. Conventional migration based on the depth extrapolation of the upgoing field fails to recognize multiple reflections shown in the right-hand figure of Fig. 1 for what they are. As a result they will give rise to fictitious reflecting boundaries in the migrated section. During migration, multiple waves produced in the top layer and presented in the observed field, 'pass' into the lower layer where, in reality, they cannot penetrate. In contrast, depth extrapolation of the total field based on analytic continuation potentially can restore the true process of wave propagation. Multiples are understood as such and hence, cannot generate fictitious reflecting boundaries as long as the analytic continuation is done accurately.

Fig. 2 shows snapshots of the analytically continued wavefield at specified moments in time. Input consisted of the data shown at the right of Fig. 1. At the latest time shown (150 ms), M is a multiple reflection from within the shallow layer. As time decreases the spatial structure of the calculated wavefield is simplified substantially and is free of multiple waves for times of 50 ms and less.

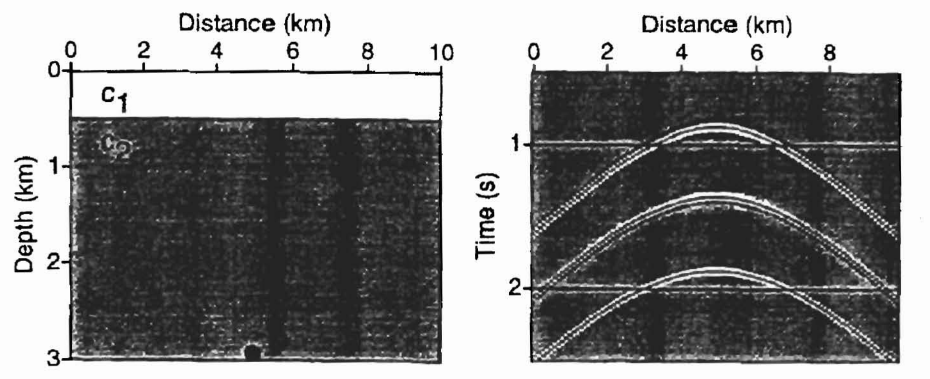

Fig. 1. (Left) Point source in half-space beneath surface layer. (Right) Synthetic zero-offset data, including two multiples.

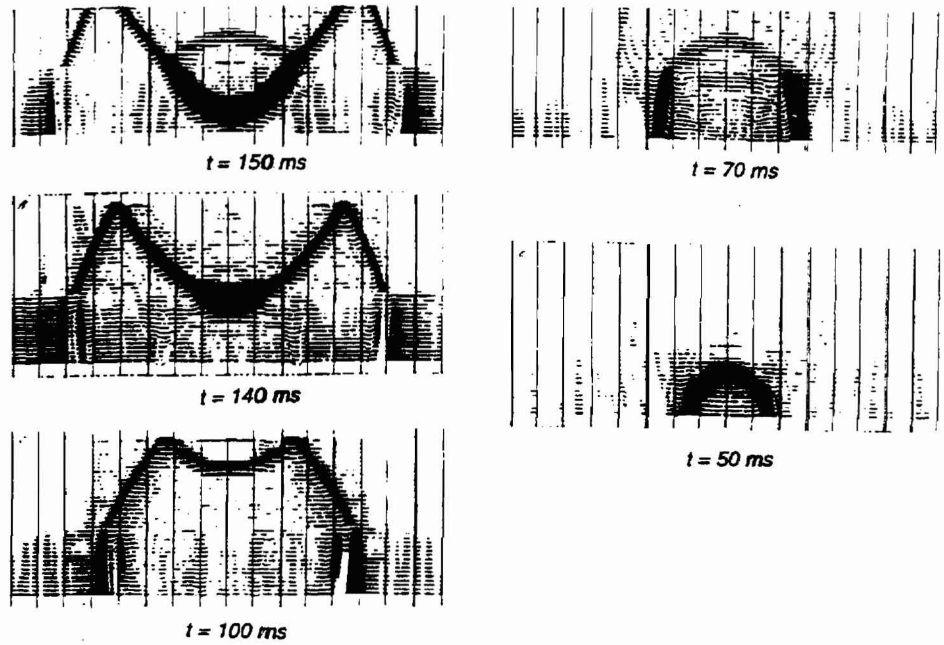

Fig. 2. Snapshots of the analytically continuated wavefield at different moments of time.

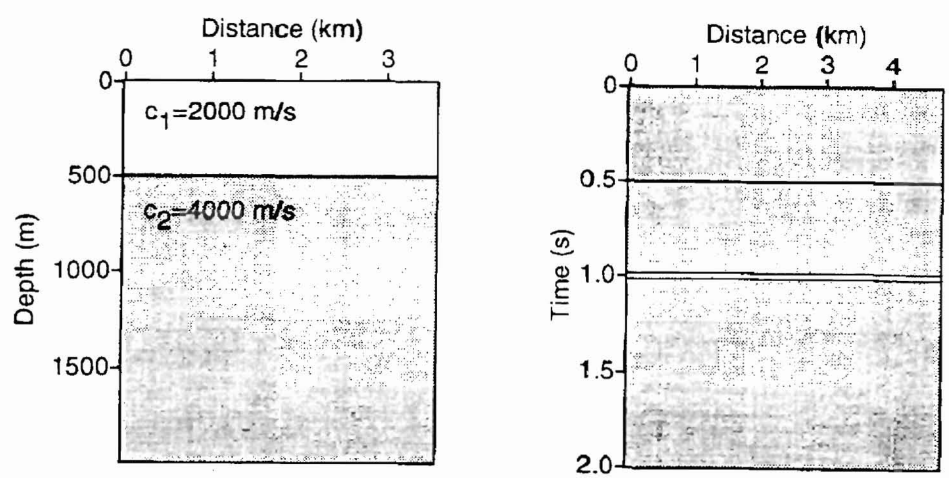

Fig. 3. (Left) Simple two-layer model with a free surface. (Right) Zero-offset data obtained from a simulated seismic reflection experiment containing a primary and one multiple reflection.

Fig. 3 shows a two-layer model with a free surface and zero-offset data. Performing conventional phase-shift migration and migration by analytic continuation resulted in Fig. 4. Conventional migration shows the false reflector caused by the presence of the multiple reflections. Migration by analytic continuation through the two-layer model gives the correct image of the section. As a final example, Figs. 5 and 6 show the modeling and migrated results of a three-layer model with a free surface and a syncline beneath the first layer. Conventional migration again shows the false reflector caused by the presence of the multiple reflections. Migration by analytic continuation gives the correct image of the section, because most energy has been migrated to the position of primary. This is caused by interference of the upgoing and downgoing wavefields (conventional phase-shift migration considers only the upgoing wavefield).

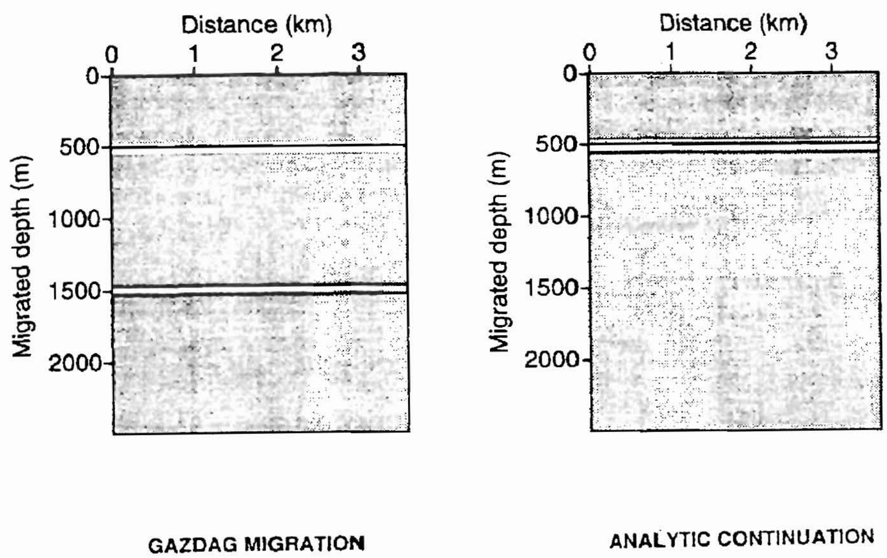

Fig. 4. (Left) Depth-migrated data using conventional Gazdag migration. (Right) Depth-migrated data using migration by analytic continuation.

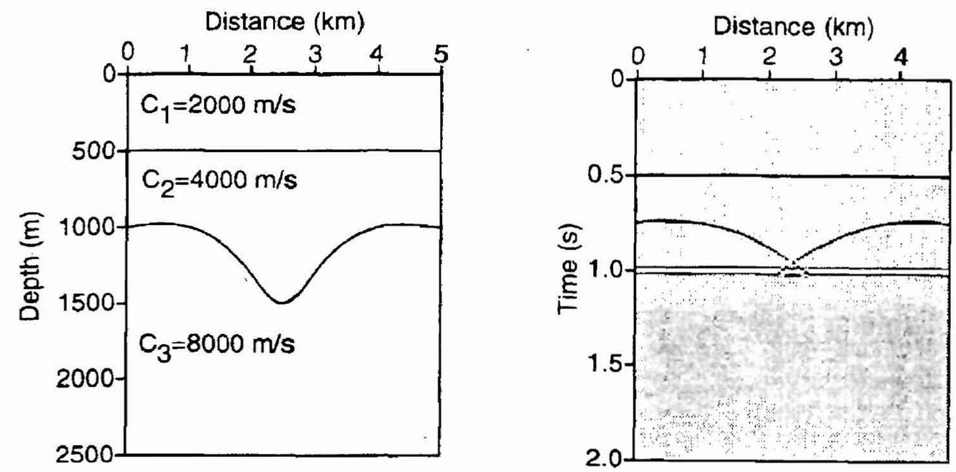

Fig. 5. (Left) Three-layer model with a free surface and a syncline below the first layer. (Right) Zero-offset data obtained from a simulated seismic reflection experiment containing the primaries and one multiple reflection.

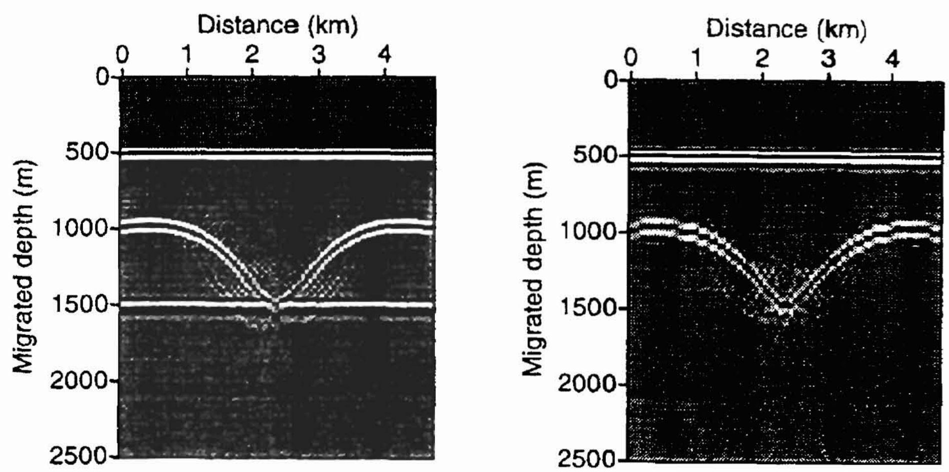

Fig. 6. (Left) Depth-migrated data using conventional Gazdag migration. (Right) Depth-migrated data using migration by analytic continuation.

CONCLUSIONS

Migration by analytic continuation allows reconstruction of the total distribution of the seismic waves inside the earth. Implementation of this method is no more complicated than that of conventional migration (Stolt or Gazdag) in the frequency-wavenumber domain. The developed method can be applied to geological models with strong velocity contrasts. The method offers the opportunity to treat not only those multiple reflections related to the perfectly reflecting free surface, but also those from strong velocity boundaries within the subsurface. However, actually treating such multiplies will involve complications not addressed in this paper. A layer-by-layer effort that will undoubtedly be highly sensitive to the interpretation of layer boundary depth will be required. The quality of the treatment of multiples by this migration technique will, thus, be degraded by the errors in the estimation of the background velocity as well as the acoustic-impedance discontinuity and location of the interfaces, that produce the multiples. This problem is of fundamental importance to the method and requires further special treatment.

Another important problem is related to the fact that the earth is never exactly horizontally layered. So, it is necessary to find the way to generalize the developed technique in order to account for a laterally varying background model. In principle, the solution of this problem can be found using the same technique of the Green's function decomposition, that we have used in this paper (see the Appendix). Actually, the problem can be solved even for elastic wave equations, using the asymptotic expression of the elastic Green's tensor for high frequency, obtained by Cohen (1988) for a completely *inhomogeneous* medium. However, the detailed analysis of this inhomogeneous model requires a lot of calculation and will be the subject of a separate paper.

Therefore much work remains ahead.

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APPENDIX

1-D GREEN'S FUNCTION IN THE SOLUTION OF THE BOUNDARY VALUE PROBLEM

In the simplest case, when $V(z) = V = const$ and, correspondingly, $C(z) = C = const$, the Green's function can be expressed in the form

$$g(z,\zeta) = -(C/2i\omega)\exp\{\pm(i\omega/C)(z - \zeta)\} \quad , \tag{A-1}$$

where we use the sign + for $z > \zeta$ and the sign − for $z < \zeta$. In the general case of variables $V(z)$ and $C(z)$, it is difficult to construct the direct analytical solution for the Green's function. However, we can effectively use the high-frequency asymptotic solution (the so-called WKBJ approximation) which, in the case of the one-dimensional Green's function, can be written, following Bleistein (1984), as

$$g(z,\zeta) \sim -\sqrt{\{C(\zeta)C(z)\}}/2i\omega \exp[\pm i\omega \int_{\zeta}^{z} \{d\xi/C(\xi)\}] \quad , \qquad \text{(A-2)}$$

where as in equation (A-1) we use the sign + for $z > \zeta$ and the sign − for $z<\zeta$. For constant velocity equation (A-2) reduces to (A-1). For the construction of the WKBJ approximation we have to neglect the evanescent waves so that according to (19) and (20) we suppose that

$$C^2(z) > 0 \quad .$$

Now, if we know the field's values on the bottom of layer $z = h_1$, we can use equation (8) to solve the boundary value problem - recalculation of the field u inside the earth. But the field is not initially known at depth. To overcome this difficulty, let us represent the Green's function $g(h_1,\zeta)$ in the form

$$g(h_1,\zeta) \sim a(h_1)b(\zeta) \quad , \qquad \text{(A-3)}$$

where

$$a(z) = -\{\sqrt{C(z)} / 2i\omega\}\exp[i\omega \int_{0}^{z} \{d\xi/C(\xi)\}] \quad ,$$

$$b(\zeta) = \sqrt{C(\zeta)} \exp[-i\omega \int_{0}^{\zeta} \{d\xi/C(\xi)\}] \quad . \qquad \text{(A-4)}$$

Everywhere inside the first layer the function a(z) is the high-frequency asymptotic solution of the Helmholtz equation

$$La(z) = 0 \quad . \qquad \text{(A-5)}$$

Therefore, from Green's theorem (6), we have

$$\tilde{u}(h_1)g'(h_1,\zeta) - g(h_1,\zeta)\tilde{u}'(h_1) = \tilde{u}(0)a'(0)b(\zeta) - a(0)b(\zeta)\tilde{u}'(0) \quad . \qquad \text{(A-6)}$$

Substituting (A-6) into (8) we have

$$\tilde{u}(\zeta) = -\tilde{u}(0)[a'(0)b(\zeta) - g'(0,\zeta)] + \tilde{u}'(0)[a(0)b(\zeta) - g(0,\zeta)] \quad . \qquad \text{(A-7)}$$

According to (A-2) and (A-4)

$$g(0,\zeta) \sim -[\sqrt{\{C(\zeta)C(0)\}} / 2i\omega] \exp[i\omega\phi(\zeta)] \quad , \qquad \text{(A-8)}$$

$$a(0)b(\zeta) \sim -[\sqrt{\{C(\zeta)C(0)\}} / 2i\omega\} \exp[-i\omega\phi(\zeta)] \quad , \qquad \text{(A-9)}$$

where

$$\phi(\zeta) = \int_{0}^{\zeta} \{d\xi/C(\xi)\} \quad . \qquad \text{(A-10)}$$

The leading order terms for $g'(0,\zeta)$ and $a'(0)b(\zeta)$ are thus

$$g'(0,\zeta) \sim (1/2)\sqrt{\{C(\zeta)/C(0)\}}\ \exp[i\omega\phi(\zeta)] \quad , \tag{A-11}$$

$$a'(0)b(\zeta) \sim -(1/2)\sqrt{\{C(\zeta)/C(0)\}}\ \exp[-i\omega\phi(\zeta)] \quad . \tag{A-12}$$

Substituting equations (A-8), (A-9) and (A-11), (A-12) into (A-7) and taking into consideration the boundary conditions (5) we obtain

$$\tilde{u}(\zeta) = \tilde{u}^0\sqrt{\{C(\zeta)/C(0)\}}\ \cos[\omega\phi(\zeta)] + \tilde{u}^1[\sqrt{\{C(\zeta)C(0)\}}\ /\ \omega]\ \sin[\omega\phi(\zeta)]\ . \tag{A-13}$$

Also in the high frequency approximation, differentiation of the last equation gives the vertical field derivative inside the first layer

$$\tilde{u}'(\zeta) = \tilde{u}^0[-\omega/\sqrt{\{C(\zeta)C(0)\}}]\sin[\omega\phi(\zeta)] + \tilde{u}^1\sqrt{\{C(0)/C(\zeta)\}}\cos[\omega\phi(\zeta)]\ . \tag{A-14}$$

Thus, using the formulae (A-13) and (A-14) along with proper boundary conditions on the interfaces we can analytically continue the wave field u from the surface of the earth through a variable background medium to any internal point of the earth's interior.