Electromagnetic scattering of large structures in layered earths using integral equations

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Abstract. An electromagnetic scattering algorithm for large conductivity structures in stratified media has been developed and is based on the method of system iteration and spatial symmetry reduction using volume electric integral equations. The method of system iteration divides a structure into many substructures and solves the resulting matrix equation using a block iterative method. The block submatrices usually need to be stored on disk in order to save computer core memory. However, this requires a large disk for large structures. If the body is discretized into equal-size cells it is possible to use the spatial symmetry relations of the Green’s functions to regenerate the scattering impedance matrix in each iteration, thus avoiding expensive disk storage. Numerical tests show that the system iteration converges much faster than the conventional point-wise Gauss-Seidel iterative method. The numbers of cells do not significantly affect the rate of convergency. Thus the algorithm effectively reduces the solution of the scattering problem to an order of $O(N^2)$, instead of $O(N^3)$, as with direct solvers.

1. Introduction

Electromagnetic scattering of conductivity structures in geophysical applications using the integral equation method goes back to the early 1970s when Hohmann used a volume integral equation approach to calculate the responses of Earth conductivity models [Hohmann, 1971, 1975; Raiche, 1974; Weidelt, 1975]. This method has undergone continuous developments in the last two decades. An early concern among practitioners of the method was the computational expense of the method, which limited its use to simple or small models. A concerted effort has made to address this problem. For scatterers with two planes of symmetry Tripp and Hohmann [1984] and Tripp [1990] presented a group theoretic block diagonalization method for arbitrary sources of excitation which reduces the scattering impedance matrix to a block diagonal matrix, with subsequent reduction of storage by a factor of 16 and the matrix factorization time by a factor of 64.

More recently Xiong [1992b] introduced the method of system iteration, where a scatterer is divided into substructures and direct matrix inversion is applied to each substructure, while the mutual interactions among the substructures introduce equivalent sources. This technique greatly reduces the matrix factorization time and storage requirements. This block iterative method has also been combined with the block diagonalization method for symmetric models [Xiong and Tripp, 1995]. Using the lateral homogeneity of a layered Earth Xiong and Tripp [1993] developed a spatial symmetry reduction scheme which drastically reduces the computation time for forming the scattering impedance matrix by identifying and reducing the redundancy of calculating the scattering matrix elements. This last scheme permits enhanced use of the previously mentioned algorithmic improvements, as we will show.

The method of system iteration is applicable to arbitrary scatterers with arbitrary discretizations. But it needs to store all the subblock matrices on a disk. This does reduce the memory requirement but requires a large disk for large models. Also, to...
access the disk takes a considerable amount of computer time and appears to be a bottleneck for computers with fast processors. While the method of system iteration is applicable regardless of the discretizations of the substructures, equally discretized structures give new vitality to this method. If all the substructures are discretized into equal-size cells, as required by the spatial symmetry reduction, we can generate the scattering impedance matrix in each iteration without storing it. Thus we only need to store the nonidentical entries of the impedance matrix in the computer memory. This will allow us to run tens of thousands of cells on workstation-type computers.

Constructing techniques with those in the electrical engineering literature can be instructive. Many rapid algorithms have emerged recently for the solution of scattering problems of electrically large bodies in free space using integral equations [Sarkar, 1991; Sarkar et al., 1986; Rokhlin, 1990; Canning, 1993; Chew and Lu, 1993; Chew, 1992]. Although these methods perform well for scatterers in whole spaces they cannot be adapted to arbitrary electric media without losing certain advantages.

In geophysical exploration we often assume the earth to be stratified with conductivity varying only in the vertical direction. While the conjugate gradient fast Fourier transformation (CG-FFT) method [Sarkar, 1991; Sarkar et al., 1986] is well adapted to calculating the response of a scatterer in a whole space due to the convolutional nature of the integral equations [Bajarski, 1972], the volume integral equation for a body in stratified media does not have a convolutional form in the vertical direction, and thus FFT types of spectral iterative methods based on convolutional integrals either cannot be applied to stratified media or incur severe limitations and reduced efficiency. Also, the Toeplitzlike matrix approach [Mackay and McCowen, 1988] will suffer from decreased efficiency because the matrix elements relating cells in the vertical direction are influenced by the stratification.

In what follows we shall briefly introduce the method of Xiong [1992b] and Xiong and Tripp [1993] and demonstrate the performance of a hybrid of these two methods. We include a convergence analysis of a practical model.

2. The Integral Equation Formulism and the Method of System Iteration

Let us consider a three-dimensional (3D) model embedded in a layered Earth which is assumed to be isotropic with a complex conductivity \( \sigma_n = \sigma_n' + i\omega\varepsilon_0 \), as shown in Figure 1. The scattering current, denoted by \( \mathbf{J}_s \), within an arbitrary 3D structure embedded in the host medium is defined as

\[
\mathbf{J}_s = \Delta \sigma \mathbf{E},
\]

with \( \Delta \sigma = \sigma - \sigma_n \), where \( \sigma \) is the conductivity of the structure, and \( \mathbf{E} \) is the total electric field in the structure. The scattering current \( \mathbf{J}_s \) within a structure \( V \) satisfies the following Fredholm integral equation of the second kind:

\[
\mathbf{J}_s(r')/\Delta \sigma = - \int_V \mathbf{G}_E^E(r|r') \cdot \mathbf{J}_s(r') \, dv' = \mathbf{E}_n(r),
\]

where \( \mathbf{E}_n \) is the incident electric field in the absence of the 3D scatterer, and \( \mathbf{G}_E^E \) is the electric Green's tensor [Weidelt, 1975; Stoyer, 1977; Ali and Mahmoud, 1979; Tang, 1979; Wannamaker et al., 1984; Xiong, 1989, note that the \( k/r \) in (28) and (29) should be \( u/r \) and \( v/r \), respectively]. Once \( \mathbf{J}_s \) is found from (2) the electromagnetic fields everywhere can be determined by the integration of the scattering current with the corresponding Green's functions.
Equation (2) is discretized by dividing the structure \( V \) into \( M \) cells, using the method of moments with pulse basis functions and \( \delta \) weights. This yields the matrix equation,

\[
[\Gamma] [J_s] = [E_n].
\]

The coefficient matrix \( \Gamma \) in (3) is referred to as the scattering impedance matrix. The size of \( \Gamma \) is \( N \) by \( N \) where \( N = 3 \times M \), since we have 3 components for each vector quantity.

Following Xiong [1992a], let us consider the model shown in Figure 1. We may obtain an approximate solution for the scattering currents in the block defined by \( \sigma_1 \) if we consider the influence of other blocks as external source excitations in addition to the incident field. All four blocks may be treated in a similar fashion. We can divide a structure \( V \) into many smaller substructures, \( V_1, V_2, \ldots, V_m \), and rewrite (2) as follows:

\[
J_s(r)/\Delta \sigma_1 - \int_{V_1} G(r|r') \cdot J_s(r') \, dv' = E_n(r) + \sum_{j,j' \neq i} \int_{V_j} G(r|r') \cdot J_s(r') \, dv'.
\]

We are then able to solve for the scattering currents in all of the substructures by an iterative approach. Equation (4) shows that we consider the mutual interactions among the substructures to be equivalent exciting sources to the \( i \)th substructure.

From the viewpoint of matrix theory, (4) is equivalent to partitioning the scattering impedance matrix \( \Gamma \) into many block matrices, \( \Gamma_{ij} \), and solving the matrix equation using a block iterative method. Thus we have

\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1m} \\
\Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_{m1} & \Gamma_{m2} & \cdots & \Gamma_{mm}
\end{bmatrix}
\begin{bmatrix}
J_{s1} \\
J_{s2} \\
\vdots \\
J_{sm}
\end{bmatrix}
= \begin{bmatrix}
E_{n1} \\
E_{n2} \\
\vdots \\
E_{nm}
\end{bmatrix},
\]

and

\[
[\Gamma_{si}] [J_{si}^{k+1}] = [E_n] - \sum_{j=1}^{i-1} [\Gamma_{sj}] [J_{sj}^{k+1}] - \sum_{j=i+1}^{m} [\Gamma_{sj}] [J_{sj}^{k}],
\]

where \( J_{si}^{k+1} \) is determined by matrix inversion. The diagonal matrices \( \Gamma_{ii} \) can be factored using \( UDU^T \) or \( LU \) factorization, depending on whether or not they are symmetric. The factored matrices are then stored for use in the system iteration. To start the iterative process we simply choose zero for the initial values of \( J_s \). As we will see later in Figure 3, the solution converges very rapidly at the initial stage, which means that the choice of initial scattering current value does not affect the computation very much. This block iterative method is a generalization of the conventional point Gauss-Seidel method. The iterative procedure is terminated if

\[
\frac{\|J_{s}^{k+1} - J_{s}^{k}\|}{\|J_{s}^{k}\|} \leq 10^{-4}.
\]

The above stopping criterion is convenient. It results in an accuracy of about 0.1% in the secondary electric and magnetic fields at receiver sites on the Earth’s surface, which is sufficient in our studies.

This system iteration method reduces computer memory requirements compared to the direct solution of (5), since only the diagonal submatrices corresponding to each substructure need to be factorized directly. Since the unknowns in each subsystem are evaluated directly, the iterative algorithm (6) converges very rapidly and saves matrix solution computation time. It is also more stable than the direct inversion of the entire matrix.

The formulism of (5) and (6) is rather general and does not require any special features of the matrix system. As long as the iterative algorithm converges, it should have the aforementioned advantages. In particular, this method of system iteration does not require a uniform discretization as with the iterative algorithms based on FFT. However, if the structure is discretized into uniform, or equal-size cells we can avoid the storage of the matrix elements using a spatial symmetry reduction technique to recompute the them in each iteration, as discussed below.

### 3. System Iteration Using Spatial Symmetry Reductions

The method of system iteration utilizes the full scattering impedance matrix and need to store the submatrices in (5) on a disk. This usually requires a large disk for large models. However, if the body is discretized into equal-dimensional cells we can take advantage of the spatial symmetry property of the space in lateral directions and regenerate the impedance matrix in each iteration without storing it.
The spatial symmetry of the layered Earth in lateral directions permits a reduction of the computation for forming the impedance matrix [Xiong and Tripp, 1993]. This reduction occurs when a conductivity structure is discretized into equal-size cells, because the lateral homogeneity of the space and the symmetry property of the Green’s functions partition the model cells into equivalent classes of cells which give either identical entries in the impedance matrix or entries that differ only in the sign. If the structure is first divided into equal-size basic cells, and some odd numbers of the basic cells are combined to form secondary, bigger cells where the scattering currents and other field quantities may be assumed to be constant, the symmetry-induced computational reduction is also valid, while the dimension of the linear system can be kept as low as possible. For equal dimensional cells the reduction factor depends mostly on the number of cells in the lateral directions, $M_x$ and $M_y$, following the formula

$$\frac{M_x M_y M_z + 1}{M_z + 1} = M_x M_y,$$

for large $M_z$, where $M_z$ is the number of cells in the vertical direction. For bodies in a whole space the factor of reduction is simply $(M_x M_y M_z + 1)/2 = (M + 1)/2$.

Due to the high efficiency in forming the matrix $\Gamma$ once the nonidentical elements are computed, we can compute the submatrices in (5) during each iteration and thus avoid data transfer bottlenecks. In this way we can solve for tens of thousands of unknown cell-scattering current values on workstation-type computers. The computation time ratio of calculating the submatrices to reading the submatrices from a disk is far less than 1 for general models. For structures with two planes of symmetry under plane wave excitations, where the total scattering matrix is reduced to one fourth the dimension of the original one by some symmetry operations, computing the submatrices is still slightly more efficient than reading them. If the group theoretic block diagonalization method is used for arbitrary sources of excitation, the computation time for generating the submatrices exceeds the time needed to read the submatrices from a disk by a factor of about 2. This is due to the fact that the block diagonalization procedure needs all the elements of the matrix corresponding to the four quadrants but only the submatrix for one quadrant is produced. However, the computation time ratio of computing the submatrices with block diagonalization operations to reading the submatrices from the disk may be improved on super computers where computation is much more efficient than accessing to disks.

In order to capitalize on the rapid convergency of the method of system iteration, we still factorize the diagonal submatrices and store them on disk. Only the off-diagonal submatrices are recomputed at each iteration. The diagonal submatrices constitute only a fraction of the total impedance matrix, and they can be made smaller and smaller by increasing the number of substructures. If we divide the model into $m$ substructures with an equal number of cells in each of the substructures, then the disk storage requirement is $N^2/m$, where $N$ is the dimension of the total impedance matrix. The adjacent elements of the matrix have a great influence on the convergence of an iterative scheme. Since the system iteration method inverts the impedance matrices in each substructure, or the diagonal submatrices, directly, the convergence of the scheme of (6) is very rapid. Thus storing the factorized diagonal matrices facilitates the rapid convergence of the scheme.

As will be shown by the numerical example, the number of iterations required for convergence is largely independent of the numbers of cells of a structure. Thus the computation time for the solution of the matrix equation by (6) is $O(N^2)$ instead of $O(N^3)$ as with direct solvers.

If the structures are discretized into equal-size-based cells as described by Xiong and Tripp [1993], we can still use the method of system iteration and spatial symmetry reduction without storing the off-diagonal submatrices, though the efficiency may be decreased since more computation is involved in regenerating the matrix elements.

4. Convergence of the Scheme

The convergence of the system iteration using (6) is in general much more rapid than the conventional point-wise Gauss-Seidel iterative method. The empirical rule of $2m^2$ for convergence as observed by Xiong [1992b] is a very rough estimation. Though the convergence rate depends on many factors, including the discretization and the conductivity contrast of the models, the system iteration usually converges much faster. Also the number of substructures does not greatly affect the convergence rate. This may be explained by the fact that the
convergence rate of an iterative method depends mostly on the coupling among the adjacent matrix elements. Since the method of system iteration directly solves the diagonal submatrices whose elements are strongly coupled together, the convergence of (6) is much more rapid than conventional point-wise iterative methods.

To demonstrate the convergence of the system iteration, we shall compute a large model. Our model, shown in Figure 2, is one of the 3D models for world wide comparisons of electromagnetic modeling algorithms proposed by M. S. Zhdanov for the international Comparison of Modeling of Electromagnetic Induction (COMMEMI) project [Zhdanov et al., 1990]. The model consists of two conductivity blocks of 1 and 100 Ωm. Both are 40 × 20 × 10 km\(^3\) in dimension and outcrop in a three-layered Earth with resistivities of 10, 100, and 0.1 Ωm. The upper two layers are 10 km and 20 km thick, respectively. The frequency is 0.1 Hz.

Table 1 presents the number of iterations with different numbers of substructures for the criterion (7) to be satisfied for a total of 500 and 1000 cells, with each of the two bricks being discretized into 5 × 5 × 10 cells and 10 × 5 × 10 cells. The electric field of the excitation wave was polarized in the x direction. The 250 substructures for the 500 cells and the 250 and 500 substructures for the 1000 cells are slightly nonuniformly divided into the substructures due to the automatic routine that divides the bodies into substructures. Other substructures shown in Table 1 are uniform subblocks that have the same number of cells in each case. Table 1 shows that the convergence rate of the solutions is not highly dependent on the number of substructures if the number of substructures is not too high. However, the number of iterations greatly increases as the number of substructures approaches the total number of cells. The convergence rate of the system iteration is not highly dependent on the total number of cells either. Table 2 shows the numbers of iterations required for convergence for discretizations with a total of 5 × 10 × 10 = 500, 10 × 10 × 10 = 1000, 10 × 20 × 10 = 2000, 10 × 20 × 20 = 4000, 15 × 28 × 20 = 8400, 15 × 40 × 28 = 16800, and 20 × 40 × 40 = 32000 cells. There were 10 substructures for 500, 1000, and 2000 cells; 20 substructures for 4000 cells; 40 substructures for 8400 cells; 80 substructures for 16,800 cells; and 160 substructures for 32,000 cells. The corresponding number of cells in each substructure was 1 × 5 × 10 = 50, 2 × 5 × 10 = 100, 2 × 10 × 10 = 200, 2 × 10 × 10 = 200, 3 × 14 × 5 = 210, 3 × 10 × 7 = 210, and 2 × 10 × 10 = 200, respectively. With different

![Diagram of the conductivity structure](image)

**Figure 2.** The conductivity structure consists of two blocks with resistivities of 1 Ωm and 100 Ωm outcropping in a three-layered Earth with resistivities of 10 Ωm, 100 Ωm, and 0.1 Ωm. The upper two layers of the Earth each have a thickness of 10 km and 20 km. The two blocks are each of dimensions 40 × 20 × 10 km\(^3\).
discretizations we observed similar results. If the number of iterations for convergence is strictly independent of the number of cells, the computation time for solving the matrix equation reduces to $O(N^2)$ instead of $O(N^3)$, as with direct solvers. As a comparison we also show in Table 2 the numbers of iterations for convergence using the conventional point-wise Gauss-Seidel iterative method. With the point-wise Gauss-Seidel method the number of iterations required for convergence is not a monotonic function of the number of cells. For example, 500 cells requires more iterations than does 1000 cells, and 8400 cells requires fewer iterations than do 4000 or 2000 cells. This is due to the fact that the point-wise Gauss-Seidel method is very sensitive to the condition number of the impedance matrix. The point-wise Gauss-Seidel method requires in general more than five times the number of iterations for convergence than the block-wise Gauss-Seidel method of (6). For 4000 cells the point-wise Gauss-Seidel method requires more than 30 times the number of iterations for convergence than the system iteration. These results also indicate the superiority of our method to other point-wise iterative methods, since our earlier studies have shown that the Gauss-Seidel method is the best iterative method for our scattering problems, because the impedance matrix is dominated by the diagonal elements. Though the point-wise Gauss-Seidel method is less efficient than the system iterative method, it is much more efficient than any direct solvers for this problem. For the 4000 cells case the solution converged in 517 iterations using the point-wise Gauss-Seidel method, which is the slowest example shown in Table 2. However, this took only about 1/8 of the computation time to solve the matrix equation than would be incurred by Gaussian elimination method, since the solution converged in about 1/8 of the number of iterations, that is, $(4000 \times 3)/3 = 4000$, for which the iterative algorithm takes about the same computation time as Gaussian elimination method.

From Table 1 we see that the numbers of iterations required for convergence increase slightly with the numbers of substructures until the numbers of substructures approach the total numbers of cells. As the number of substructures approaches the total number of cells, the system iteration almost reduces to conventional Gauss-Seidel iteration. But there are at least three unknowns in each substructure (each cell); thus the system iteration still converges much faster than conventional point-wise Gauss-Seidel iteration. The computation time depends mostly on the number of iterations, since the computation time for the factorizations of the diagonal submatrices decreases quadratically with the number of substructures. The storage requirement for the diagonal submatrices decreases linearly with the number of substructures. To minimize the storage requirement, it is advantageous to have as many substructures as possible. According to Tables 1 and 2, there appear to be no strict rules for the division of substructures as long as there are at least 10 cells per substructure.

Figure 3 shows the relative differences of the solutions for two successive iterations $||J_s^{k+1} - J_s^k||/||J_s^{k+1}||$, for the system iteration method and the point-wise Gauss-Seidel iteration method for the 16,800 cells case. The rapid decrease of the relative differences of the solutions with system iteration illustrates the efficiency of the system iteration method. Figure 4 shows the total CPU times on a SUN SPARCstation 10/30 for solving the scattering problem for this model with the excitation electric field polarized in the x direction. Since the computation time is overwhelmingly dominated by the solution of the matrix systems (the other parts of the computations took only a couple of percent of the total CPU time), Figure 4 in fact reflects the computation time for solving the matrix equations. From Figure 4 we see that for large numbers of cells the CPU time for the system iteration is effectively reduced to an order of $O(N^2)$, while the CPU time for the point-wise iteration is unpredictable and in general more than 5 times the CPU time of the system iterations. Let us compare with direct solvers for 32,000 cells. A direct solver, say, Gaussian elimination, would need a memory for $(3 \times 32000)^2$ complex words, or 73.728 Gigabytes in single precision, and would take a com-

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5. Concluding Remarks

The method of system iteration offers an efficient way to model large conductivity structures which permits more flexibility in discretization, since it...
does not assume any special features for the scattering impedance matrix. Thus this method is very useful for background electric media other than whole spaces. With equal-size cells we can drastically reduce the formulation time and storage requirements of the impedance matrices using the spatial symmetry relations of the Green’s functions and markedly expand the power of the method of system iteration. However, it is often true that the equal-size cells do not represent the field quantities efficiently, especially at the center or near the edge of large block structures. With nonequal-dimensional discretizations, many algorithms are not applicable. Fortunately, we still can use the method of system iteration, and if we utilize the equal-size-based cells as described by Xiong and Tripp [1993], we still can make use of the spatial symmetry reductions and solve the matrix equation with the method of system iteration without storing the off-diagonal submatrices.

The studies presented in this paper are conducted in the low frequency range where electromagnetic waves are diffusive in nature. For frequencies above 10 MHz the matrix system becomes more and more ill conditioned, and the above discussed method may be slow or fail to converge. Further study will be necessary to extend the method of system iteration for the modeling of dielectric models.

Acknowledgment. This work was sponsored by the United States DOE/OBES through grant number DE-FG03-93ER14313.

References


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(Received February 9, 1994; revised February 17, 1995; accepted February 23, 1995.)