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Three-dimensional quasi-linear electromagnetic inversion

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Abstract. One of the most challenging problems of electromagnetic (EM) geophysical methods is developing three-dimensional (3-D) EM inversion techniques. This problem is of utmost importance in practical applications because of the 3-D nature of the geological structures. The main difficulties in 3-D inversion are related to (1) limitations of 3-D forward modeling codes available and (2) ill-posedness of the inversion procedures in general. The multidimensional EM inversion techniques existing today can handle only simple models and typically are very time consuming. We developed a new approach to a rapid 3-D EM inversion. The forward scattering problem is solved using a new quasi linear (QL) approximation of the existing integral equation algorithms, developed for various sources of excitation. The QL approximation for forward modeling is based on the assumption that the anomalous field is linearly related to the normal field in the inhomogeneous domain by an electrical reflectivity tensor. We introduce also a modified material property tensor which is linearly proportional to the reflectivity tensor and the complex anomalous conductivity. The QL approximation generates a linear equation with respect to the modified material property tensor. The solution of this equation is called "a quasi-Born inversion". We apply the Tikhonov regularization for the stable solution of this problem. The next step of the inversion includes correction of the results of the quasi Born inversion: after determining a modified material property tensor, we use the electrical reflectivity tensor to evaluate the anomalous conductivity. Thus the developed inversion scheme reduces the original nonlinear inverse problem to a set of linear inverse problems, which is why we call this approach "a QL inversion". Synthetic examples (with and without random noise) of inversion demonstrate that the algorithm for inverting 3-D EM data is fast and stable.

Introduction

During the past decade, considerable advances have been made in the forward modeling of electromagnetic (EM) fields [*Tripp*, 1990; *Wannamaker*, 1991; *Xiong*, 1992; *Xiong and Tripp*, 1993; *Druskin and Knizhnerman*, 1994].

In contrast to the rapid developments in forward modeling, progress in solving the EM inverse problem in multiple dimensions has been much slower. This is due to the fact that inverse algorithms in general require more effort both in their theoretical development as well as in computational resources. The literature on EM inversion deals primarily with one- or two-dimensional inversions. Three-dimensional (3-D) EM inversion is only a recent adventure which has attracted relatively few workers [*Eaton*, 1989; *Madden and Mackie*, 1989; *Smith and Booker*, 1991; *Xiong and Kirsh*, 1992; *Tripp and Hohmann*, 1993; *Torres-Verdin and Habashy*, 1994]. However, the demand for the development of three-dimensional inverse algorithms continues to be strong enough, as one could see, for example, at the Progress in Electromagnetic Research Symposium (PIERS) in July 1995 [*Torres-Verdin and Habashy*, 1995b; *Xie and Lee*, 1995; *Newman and Alumbaugh*, 1995].

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There exist several good examples of the application of linearization methods to two-dimensional EM inverse problems [*Weidelt*, 1975; *Jupp and Vozoff*, 1977; *Oristaglio and Worthington*, 1980; *Zhdanov and Golubev*, 1983]. For example, Zhdanov and Varentsov [1983] introduced a method of tightening contours, or tightening surfaces for the determination of the shape of an inhomogeneity (see also *Berdichevsky and Zhdanov* [1984]). However, as determined by Maxwell's equations, the scattered fields are non-linearly related to the conductivity of the subsurface. Therefore EM inverse problems are characterized first of all by their nonlinearity.

Another major difficulty in inversion is related to its ill-posedness, which means that the existence, uniqueness, and/or stability of solutions are in question. The inherent nonlinearity of EM problems makes the ill-posedness more severe. To overcome this difficulty, we have to apply regularization theory to obtain a stable solution of ill-posed inverse problems [*Tikhonov and Arsenin*, 1977; *Zhdanov*, 1993; *Zhdanov and Keller*, 1994]. This approach gives a solid basis upon which to construct effective inversion algorithms for the 3-D EM problem.

It is important to emphasize that regularization does not necessarily mean "smoothing" the solution. The primary purpose of the regularization is to implement a priori information in the inversion procedure. The more information we have about the geoelectric model, the more stable is the inversion. This information is used to construct the "regularizing family" of well-posed problems, which approximate the original ill-posed inverse problem [*Zhdanov and Keller*, 1994].

In this paper we develop a new approach to a rapid 3-D EM inversion. The forward scattering problem is solved using a quasi-linear (QL) modification [*Zhdanov and Fang*, 1996] of the existing integral equation algorithms, developed for various sources of excitation, including plane waves, horizontal bipoles, vertical bipoles, horizontal rectangular loops, vertical magnetic dipoles, and the loop-loop system [*Xiong*, 1992; *Xiong and Tripp*, 1993]. The QL approximation for forward modeling is based on the assumption that the anomalous field $\mathbf{E}^a$ is linearly related to the normal field $\mathbf{E}^n$ in the inhomogeneous domain by: $\mathbf{E}^a = \hat{\lambda}\mathbf{E}^n$, where $\hat{\lambda}$ is an electrical reflectivity tensor. We introduce a modified material property tensor $\widehat{\mathbf{m}}$, linearly proportional to the reflectivity tensor and the complex anomalous conductivity $\Delta\widetilde{\sigma}$. In this case the QL approximation generates a linear equation with respect to the modified material property tensor $\widehat{\mathbf{m}}$. The solution of this equation is called "a quasi-Born inversion". We apply the Tikhonov regularization for the stable solution of this problem. After determining $\widehat{\mathbf{m}}$, we use the electrical reflectivity tensor $\widehat{\lambda}$ to evaluate the complex anomalous conductivity $\Delta\widetilde{\sigma}$ (correction of the result of the quasi-Born inversion).

Note that there is some formal similarity between our approach and the inversion scheme developed by *Habashy et al* [1993] and *Torres-Verdin and Habashy* [1994, 1995a,b] and based on the extended Born approximation. However, our scheme differs principally from theirs in the following.

1. The QL approximation and the extended Born approximation have similar formal expressions but are completely different in nature. This fact is mainly reflected in the difference of determination of the scattering tensor [*Torres-Verdin and Habashy*, 1994] and the electrical reflectivity tensor [*Zhdanov and Fang*, 1996].

The extended Born approximation is based on the calculation of the internal field as the projection of the normal electric field onto a scattering tensor $\widehat{\mathbf{\Gamma}}(\mathbf{r}) : \mathbf{E}(\mathbf{r}) = \widehat{\mathbf{\Gamma}}(\mathbf{r})\mathbf{E}^n(\mathbf{r})$. The scattering tensor $\widehat{\mathbf{\Gamma}}(\mathbf{r})$ is obtained under the assumption that the total field inside the inhomogeneity is a smoothly varying function such that its gradient can be neglected to zero order regardless of the medium properties (so called "localized approximation"). That is why in Torres-Verdin and Habashy's method the total electric field can be taken outside the integral in the integral equation, which has given rise to the explicit analytical expression for $\widehat{\mathbf{\Gamma}}(\mathbf{r})$. It is shown that the scattering tensor is a nonlinear functional of the material property distribution and does not depend on the illuminating sources. This saves some computational time for multiple source calculation. However, the accuracy of the extended Born approximation is limited by the accuracy of the localized approximation.

In the QL approximation the electrical reflectivity tensor is obtained by numerical optimization technique. It is not based on the assumption that the total field is locally constant within the inhomogeneity (it does not use "localized approximation"). It is a function of anomalous material property and, in the general case, is source dependent. This increases the computational time for multiple source calculation slightly, but not too much because the most time-consuming part of the calculations is computing the

Green's function, which are the same for different source locations. The accuracy of QL approximation depends only on the degree of discretization of the anomalous structure in the process of $\widehat{\lambda}$ determination and, in principle, can be made arbitrarily high (it tends asymptotically to the accuracy of the full integral equation solution, if discretization is chosen to be fine enough).

The QL approximation can be treated as the extension of the Born approximation and the extended Born approximation. The Born approximation appears as a special case of QL approximation if electrical reflectivity tensor $\widehat{\lambda}$ is set to be equal to zero. The extended Born approximation can be obtained from QL approximation if $\widehat{\lambda} = \widehat{\mathbf{\Gamma}} - \widehat{\mathbf{I}}$.

2. The two-step linear inversion approach developed by *Torres-Verdin and Habashy* [1995a] is based on an analytical expression for the scattering tensor $\widehat{\mathbf{\Gamma}}$, that depends explicitly on the selected model of material property distribution. We do not specify the tensor $\widehat{\lambda}$ before inversion and determine $\widehat{\lambda}$ as the result of linear inversion. That is why our scheme consists of three steps: (1) determination of the modified material property tensor $\widehat{\mathbf{m}}$, (2) evaluation of the electrical reflectivity tensor $\widehat{\lambda}$, and (3) extraction of the information about the material property from $\widehat{\mathbf{m}}$ and $\widehat{\lambda}$.

3. We solve a full vector 3-D EM problem, while *Torres-Verdin and Habashy* [1994, 1995a,b] consider a scalar problem. This does not mean that *Torres-Verdin and Habashy*'s [1995a,b] approach could not be extended to deal with the full 3-D inversion problem, but this would be a completely different method from the 3-D QL inversion scheme described below in this paper.

It is important to emphasize also that we apply the Tikhonov regularization method [*Tikhonov and Arsenin*, 1977; *Zhdanov*, 1993] for stable 3-D EM inversion. The quasi-linear inverse problem is solved by a regularized gradient type method which ensures stability and rapid convergence. Synthetic examples (with and without random noise) of inversion demonstrate that the algorithm for inverting 3-D EM data is fast and stable.

A Quasi-Linear Approximation

Consider a 3-D geoelectrical model with the normal (background) complex conductivity $\widetilde{\sigma}_n$ and local inhomogeneity D with an arbitrarily varying complex conductivity $\widetilde{\sigma} = \widetilde{\sigma}_n + \Delta\widetilde{\sigma}$, that can be, in the general case, frequency dependent. We will confine ourselves to consideration of nonmagnetic media and hence assume that $\mu = \mu_0 = 4\pi \times 10^{-7} H/m$, where μ_0 is the free-space magnetic permeability. The model is excited by an electromagnetic field generated by an arbitrary source. This field is time harmonic as $e^{-i\omega t}$. Complex conductivity includes the effect of displacement currents: $\widetilde{\sigma} = \sigma - i\omega\varepsilon$, where σ and ε are electrical conductivity and dielectric permittivity. The electromagnetic fields in this model can be presented as a sum of normal and anomalous fields:

$$\mathbf{E} = \mathbf{E}^n + \mathbf{E}^a, \quad \mathbf{H} = \mathbf{H}^n + \mathbf{H}^a, \tag{1}$$

where the normal field is a field generated by the given sources in the model with the normal distribution of conductivity $\widetilde{\sigma}_n$, and the anomalous field is produced by the anomalous conductivity distribution $\Delta\widetilde{\sigma}$.

It is well known that in this model the anomalous field can be presented as an integral over the excess currents in the inhomogeneous domain D [*Hohmann*, 1975; *Weidelt*, 1975]:

$$\mathbf{E}^a(\mathbf{r}_j) = \iiint_D \widehat{\mathbf{G}}^n(\mathbf{r}_j \mid \mathbf{r})\, \Delta\widetilde{\sigma}(\mathbf{r}) \left[\mathbf{E}^n(\mathbf{r}) + \mathbf{E}^a(\mathbf{r})\right] dv, \tag{2}$$

where $\widehat{\mathbf{G}}^n(\mathbf{r}_j \mid \mathbf{r})$ is the electromagnetic Green's tensor defined for an unbounded conductive medium with the normal conductivity $\widetilde{\sigma}_n$.

The conventional Born approximation $\mathbf{E}^B(\mathbf{r}_j)$ for the anomalous field can be obtained from (2) if we assume that the anomalous field is negligibly small inside D in comparison with the normal field. In this case it can be ignored in comparison with the normal field [*Born*, 1933]:

$$\mathbf{E}^B(\mathbf{r}_j) = \iiint_D \widehat{\mathbf{G}}^n(\mathbf{r}_j \mid \mathbf{r})\, \Delta\widetilde{\sigma}(\mathbf{r})\, \mathbf{E}^n(\mathbf{r})\, dv. \tag{3}$$

Habashy et al. [1993] and *Torres-Verdin and Habashy* [1994] developed the so called extended Born approximation, which is based on the calculation of the internal field as the projection of the normal electric field (i.e., the electric field excited in the absence of conductivity inhomogeneity) onto a scattering tensor $\widehat{\mathbf{\Gamma}}(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \widehat{\mathbf{\Gamma}}(\mathbf{r})\, \mathbf{E}^n(\mathbf{r}). \tag{4}$$

It is shown that the scattering tensor does not depend on the illuminating sources and is a nonlinear functional of the anomalous conductivity distribution [*Habashy et al*, 1993; *Torres-Verdin and Habashy*, 1994].

In the paper by *Zhdanov and Fang* [1996] we presented a different approach to the solution of the EM scattering problem, which is based, however, on similar ideas and can be considered as an extension of Torres-Verdin and Habashy's method. We separate the total electric field into normal and anomalous parts and introduce an electrical reflectivity tensor which linearly transforms the normal field into an internal anomalous one. The electrical reflectivity tensor inside inhomogeneities can be approximated by slowly varying spatial functions and can be determined numerically by a simple optimization technique. We describe here this technique for completeness.

Expression (2) can be rewritten using operator notations:

$$\mathbf{E}^a = \mathbf{C}\left[\mathbf{E}^a\right], \tag{5}$$

where $\mathbf{C}\left[\mathbf{E}^a\right]$ is an integral operator of the anomalous field $\mathbf{E}^a$, $\mathbf{C}\left[\mathbf{E}^a\right] = \mathbf{A}\left[\mathbf{E}^n\right] + \mathbf{A}\left[\mathbf{E}^a\right]$, and $\mathbf{A}$ is a linear scattering operator

$$\mathbf{A}[\mathbf{E}] = \int\int\int_D \widehat{\mathbf{G}}^n\left(\mathbf{r}_j \mid \mathbf{r}\right) \Delta\widetilde{\sigma}(\mathbf{r}) \mathbf{E}(\mathbf{r})\, dv. \tag{6}$$

Equation (5) can be treated as an integral equation with respect to the anomalous field $\mathbf{E}^a$. The solution of this integral equation has to be a fixed point of the operator $\mathbf{C}$. This solution can be derived with the method of successive iterations, which is governed by the equations:

$$\mathbf{E}^{a(N)} = \mathbf{C}\left[\mathbf{E}^{a(N-1)}\right],\ N = 1, 2, 3... \tag{7}$$

It is well known that successive iterations converge, provided that the operator $\mathbf{C}$ is a contraction operator, that is if $\|\mathbf{A}\| < 1$. The Born approximation is simply the first iteration of this method, if the initial approximation $\mathbf{E}^{a(0)}$ (zero order iteration) is selected to be equal to zero ($\mathbf{E}^{a(0)} = 0$):

$$\mathbf{E}^B = \mathbf{E}^{a(1)} = \mathbf{C}[0] = \mathbf{A}\left[\mathbf{E}^n\right]. \tag{8}$$

We will obtain a more accurate approximation if we assume that the anomalous field $\mathbf{E}^a$ inside the inhomogeneous domain is not equal to zero, but is linearly related to the normal field $\mathbf{E}^n$ by some tensor $\widehat{\lambda}$:

$$\mathbf{E}^a(\mathbf{r}) \approx \widehat{\lambda}(\mathbf{r}) \mathbf{E}^n(\mathbf{r}), \tag{9}$$

which we call "an electrical reflectivity tensor".

Therefore, we use expression (9) as the zero-order approximation for the scattered field inside the inhomogeneity ($\mathbf{E}^{a(0)} = \widehat{\lambda}\mathbf{E}^n$) and calculate the first approximation as follows:

$$\begin{aligned}\mathbf{E}^{a(1)} &= \mathbf{C}\left[\widehat{\lambda}\mathbf{E}^n\right] = \mathbf{A}\left[\mathbf{E}^n + \widehat{\lambda}\mathbf{E}^n\right] \\ &= \mathbf{A}\left[\left(\widehat{\mathbf{I}} + \widehat{\lambda}\right)\mathbf{E}^n\right] = \mathbf{E}^a_{ql}.\end{aligned} \tag{10}$$

We call this approximation "a quasi linear (QL) approximation" $\mathbf{E}^a_{ql}$ for the anomalous field.

Let us rewrite equation (10) for the points inside domain D, taking into account (9):

$$\mathbf{E}^a_{ql} \approx \widehat{\lambda}\mathbf{E}^n \approx \mathbf{A}[(\mathbf{I} + \widehat{\lambda})\mathbf{E}^n]$$

$$= \int\int\int_D \widehat{\mathbf{G}}^n\left(\mathbf{r}_j \mid \mathbf{r}\right) \Delta\widetilde{\sigma}(\mathbf{r}) \left[\widehat{\mathbf{I}} + \widehat{\lambda}(\mathbf{r})\right] \mathbf{E}^n(\mathbf{r})\, dv. \tag{11}$$

This last equation provides the basis for determining $\widehat{\lambda}$. It should hold for any internal point of the domain D. In reality, of course, it holds only approximately. Also, we assume that the electrical reflectivity tensor inside inhomogeneties can be approximated by a relatively slowly varying function [*Zhdanov and Fang*, 1996]. Therefore we can use the minimum norm condition to determine $\widehat{\lambda}$:

$$\begin{aligned}&\Big\| \widehat{\lambda}(\mathbf{r}_j) \mathbf{E}^n(\mathbf{r}_j) - \\ &- \textstyle\int\int\int_D \widehat{\mathbf{G}}^n\left(\mathbf{r}_j \mid \mathbf{r}\right) \Delta\widetilde{\sigma}(\mathbf{r}) \left[\left(\widehat{\mathbf{I}} + \widehat{\lambda}\right) \mathbf{E}^n(\mathbf{r})\right] dv \Big\| \\ &= \min.\end{aligned} \tag{12}$$

After $\widehat{\lambda}$ is found, the QL approximation is calculated using expression (11). Note that the electrical reflectivity tensor $\widehat{\lambda}$ can be formally related to the scattering tensor $\widehat{\Gamma}$, introduced by Habashy et al [1993], by the simple formula

$$\widehat{\lambda} = \widehat{\mathbf{\Gamma}} - \widehat{\mathbf{I}}. \tag{13}$$

However, despite this formal similarity between the QL approximation and the extended Born approximation, they are completely different in nature. This fact is mainly reflected in the difference of determination of the scattering tensor [*Torres-Verdin and*

Habashy, 1994] and the electrical reflectivity tensor [*Zhdanov and Fang*, 1996]: Torres-Verdin and Habashy use an analytical expression to determine $\widehat{\Gamma}$, while we find the electrical reflectivity tensor by solving minimization problem (12). The analytical expression for the scattering tensor $\widehat{\Gamma}(\mathbf{r})$ is obtained by Torres-Verdin and Habashy [1994] under the assumption that the total field inside the inhomogeneity is a smoothly varying function such that its gradient can be neglected to zero order regardless of the medium properties (so called "localized approximation"). Therefore the accuracy of the extended Born approximation is limited by the accuracy of the localized approximation. In the QL approximation the electrical reflectivity tensor is obtained by a numerical optimization technique that is not based on "localized approximation". This numerical approach opens a way to impose additional restrictions on the reflectivity tensor and improves the accuracy of the QL approximation through the appropriate selection of $\widehat{\lambda}$ [*Zhdanov and Fang*, 1996]. The accuracy of QL linear approximation depends only on the degree of discretization of the anomalous structure in the process of $\widehat{\lambda}$ determination and, in principle, can be made arbitrarily high.

According to equations (2) and (10), the anomalous electromagnetic field $\mathbf{E}^a$, $\mathbf{H}^a$, is linearly related to the electrical reflectivity tensor $\widehat{\lambda}$ and the normal field $\mathbf{E}^n$ inside the inhomogeneous domain D by the integral formula

$$\mathbf{F}^a \approx \iiint_D \widehat{\mathbf{G}}^F(\mathbf{r}_j \mid \mathbf{r})\, \Delta\widetilde{\sigma}(\mathbf{r}) \left[\widehat{\mathbf{I}}+\widehat{\lambda}(\mathbf{r})\right] \mathbf{E}^n(\mathbf{r})\, dv, \tag{14}$$

where $\mathbf{F}^a$ stands for $\mathbf{E}^a$ or $\mathbf{H}^a$ observed on the surface of the Earth, $\mathbf{E}^n$ is a field generated by the given sources in the model with the normal distribution of conductivity $\widetilde{\sigma}_n$, and $\widehat{\mathbf{G}}^F(\mathbf{r}_j \mid \mathbf{r})$ stands for the electric or magnetic Green's tensor defined for an unbounded conductive medium with the normal conductivity $\widetilde{\sigma}_n$. However, this equation is nonlinear with respect to $\Delta\widetilde{\sigma}$ (because $\widehat{\lambda}$ is, in general case, a nonlinear function of $\Delta\widetilde{\sigma}$). In the next section we will discuss the method of transforming this equation into a system of linear equations.

A Quasi-Linear (QL) Inversion

A Quasi-Born Inversion

Equation (14) can be treated as a linear equation if we introduce a new tensor function,

$$\widehat{\mathbf{m}}(\mathbf{r}) = \Delta\widetilde{\sigma}(\mathbf{r}) \left[\widehat{\mathbf{I}}+\widehat{\lambda}(\mathbf{r})\right], \tag{15}$$

which we will call "a modified material property tensor". In this case, equation (14) takes the form

$$\mathbf{F}^a(\mathbf{r}_j) \approx \iiint_D \widehat{\mathbf{G}}^F(\mathbf{r}_j \mid \mathbf{r})\, \widehat{\mathbf{m}}(\mathbf{r})\, \mathbf{E}^n(\mathbf{r})\, dv. \tag{16}$$

Equation (16) is a linear one with respect to $\widehat{\mathbf{m}}(\mathbf{r})$ (while the original equation (14) was nonlinear with respect to $\Delta\widetilde{\sigma}$). Its structure is similar to that of the Born approximation for the anomalous field, if we replace the modified material property tensor $\widehat{\mathbf{m}}(\mathbf{r})$ by the anomalous conductivity $\Delta\widetilde{\sigma}(\mathbf{r})$. That is why we will call equation (16) "a quasi Born approximation". The solution of the linear equation (16) with respect to the modified material property tensor $\widehat{\mathbf{m}}(\mathbf{r})$ will be called "a quasi-Born inversion".

There are many different techniques for a solution of linear integral equation. We apply the approach based on the Tikhonov regularization method [*Tikhonov and Arsenin*, 1977] (see Appendices A and B).

The reflectivity tensor $\widehat{\lambda}$ can also be determined from the following linear equation inside the inhomogeneous domain D, as soon as we know $\widehat{\mathbf{m}}$:

$$\begin{aligned} \mathbf{E}^a(\mathbf{r}_j) &\approx \iiint_D \widehat{\mathbf{G}}^E(\mathbf{r}_j \mid \mathbf{r})\, \widehat{\mathbf{m}}(\mathbf{r})\, \mathbf{E}^n(\mathbf{r})\, dv \\ &\approx \widehat{\lambda}(\mathbf{r}_j)\, \mathbf{E}^n(\mathbf{r}_j). \end{aligned} \tag{17}$$

Once $\widehat{\mathbf{m}}$ and $\widehat{\lambda}$ are determined, it is possible to evaluate the anomalous conductivity $\Delta\widetilde{\sigma}$ from equation (15).

This inversion scheme reduces the original nonlinear inverse problem to three linear inverse problems: the first (the quasi Born inversion) for the parameter $\widehat{\mathbf{m}}$, another one for the parameter $\widehat{\lambda}$, and the third one (correction of the result of the quasi Born inversion) for the conductivity $\Delta\widetilde{\sigma}$. This approach is based on a QL approximation, so we call it "a QL inversion".

Numerical Scheme of QL Inversion

In practice, it is convenient to use some simplification, based on the division of the domain D into substructures (subdomains) $D = \cup_{k=1,K} D_k$ and the assumption that relationship (9) holds inside any substructure D_k :

$$\mathbf{E}^a(\mathbf{r}) = \widehat{\lambda}^k(\mathbf{r})\,\mathbf{E}^n(\mathbf{r}),\ \mathbf{r}\in D_k, \tag{18}$$

where the reflectivity tensor $\widehat{\lambda}^k$ depends only on the substructure's number k (note that $\widehat{\lambda}^k$ is still a function of frequency). Substituting (18) into (2), we have

$$\mathbf{F}^a(\mathbf{r}_j) = \sum_{k=1,N}\iiint_{D_k}\widehat{\mathbf{G}}^F(\mathbf{r}_j\mid\mathbf{r})\,\widehat{\mathbf{m}}^k(\mathbf{r})\,\mathbf{E}^n(\mathbf{r})\,dv. \tag{19}$$

where

$$\widehat{\mathbf{m}}^k(\mathbf{r}) = \Delta\widetilde{\sigma}(\mathbf{r})\left[\widehat{\mathbf{I}}+\widehat{\lambda}^k(\mathbf{r})\right] \tag{20}$$

is a modified material property tensor depending only on the substructure's number k and frequency.

For the numerical calculation it is more convenient to rewrite equation (19) using tensor notation:

$$F^a_\alpha(\mathbf{r}_j) = \sum_{k=1,N}\iiint_{D_k}G^F_{\alpha\beta}(\mathbf{r}_j\mid\mathbf{r})\,m^k_{\beta\gamma}(\mathbf{r})\,E^n_\gamma(\mathbf{r})\,dv; \tag{21}$$

$$\alpha,\beta,\gamma = x,y,z.$$

In (21), $F^a_\alpha(\mathbf{r})$, $E^n_\gamma(\mathbf{r})$ $(\alpha,\gamma = x,y,z)$ are the Cartesian components of the electromagnetic field vectors, $G^F_{\alpha\beta}(\mathbf{r}_j\mid\mathbf{r})$ $(\alpha,\beta = x,y,z)$ are the Cartesian components of the electromagnetic Green's tensor, and $m^k_{\beta\gamma}(\mathbf{r})$ $(\beta,\gamma = x,y,z)$ are the Cartesian components of the modified material property tensor. In all equations with summations we use the Einstein convention: the twice recurring index indicates summation over this index (for example, $\beta = x,\ y,\ z$).

Let us assume now that inside each substructure (subdomain) D_k, the conductivity is constant $\Delta\sigma(\mathbf{r}) = \Delta\sigma_k$ and the reflectivity tensor is constant $\lambda_{\beta\gamma}(\mathbf{r}) = \lambda^k_{\beta\gamma}$ (note that $\lambda^k_{\beta\gamma}$ is still a function of frequency). Therefore parameter $m^k_{\beta\gamma}(\mathbf{r})$ also depends only on the substructure's number k and frequency:

$$m_{\beta\gamma}(\mathbf{r}) = m^k_{\beta\gamma} = \Delta\widetilde{\sigma}_k\left[\delta_{\beta\gamma}+\lambda^k_{\beta\gamma}\right],\ \mathbf{r}\in D_k. \tag{22}$$

Substituting equation (22) into (21), we obtain

$$F^a_\alpha(\mathbf{r}_j) = \sum_{k=1,N}m^k_{\beta\gamma}\iiint_{D_k}G^F_{\alpha\beta}(\mathbf{r}_j\mid\mathbf{r})\,E^n_\gamma(\mathbf{r})\,dv. \tag{23}$$

Equation (17) will take the form:

$$\begin{aligned}E^a_\alpha(\mathbf{r}_j) &= \sum_{k=1,N}m^k_{\beta\gamma}\iiint_{D_k}G^E_{\alpha\beta}(\mathbf{r}_j\mid\mathbf{r})\,E^n_\gamma(\mathbf{r})\,dv\\ &= \lambda^j_{\alpha\gamma}E^n_\gamma(\mathbf{r}_j)\end{aligned} \tag{24}$$

$$\alpha,\beta,\gamma = x,y,z.$$

Equation (22) will be reduced to

$$m^k_{\beta\gamma}(\omega_l) = \Delta\widetilde{\sigma}_k(\omega_l)\left[\delta_{\beta\gamma}+\lambda^k_{\beta\gamma}(\omega_l)\right], \tag{25}$$

if we outline the dependence of $m^j_{\beta\gamma}$, $\Delta\widetilde{\sigma}_k$, and $\lambda^j_{\alpha\gamma}$ on frequency ω_l, $l = 1,2,..L$. We have to notice that equation (25) should hold for any frequency. In reality, of course, it holds only approximately. Therefore we can use the least squares method to determine the $\Delta\widetilde{\sigma}_j$ that optimally fits (25). Hence we wish to minimize

$$\begin{aligned}&\varphi(\Delta\widetilde{\sigma}_j) =\\ &\sum_{l=1}^{L}\sum_{j=1}^{N}\sum_{\beta,\gamma=x,y,z}\left[m^j_{\beta\gamma}(\omega_l)-\Delta\widetilde{\sigma}_j(\omega_l)\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)\right]^*\\ &\cdot\left[m^j_{\beta\gamma}(\omega_l)-\Delta\widetilde{\sigma}_j(\omega_l)\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)\right]\\ &= \min,\end{aligned} \tag{26}$$

where the asterisk denotes the complex conjugate value.

The solution of the minimization problem (26) (under the assumption that $m^j_{\beta\gamma}(\omega_l)$ and $\lambda^j_{\beta\gamma}(\omega_l)$ are already determined) is described in Appendix C. Real and imaginary parts of this solution give us the distribution of anomalous electrical conductivity $\Delta\sigma$ and anomalous dielectric permittivity $\Delta\varepsilon$, which can be, in general, frequency dependent:

$$\Delta\sigma_j(\omega_l) = \mathrm{Re}\Delta\widetilde{\sigma}_j(\omega_l),\ \Delta\varepsilon_j(\omega_l) = \mathrm{Im}\Delta\widetilde{\sigma}_j(\omega_l)/\omega_l. \tag{27}$$

Regularized Solution of Linear Equations

Let us rewrite equations (23) using a matrix notation:

$$\underline{\mathbf{F}} = \underline{\mathbf{G}}^F\,\underline{\mathbf{m}}. \tag{28}$$

Here $\underline{\mathbf{m}}$ is the matrix of modified material property tensor $\widehat{\mathbf{m}}$, $\underline{\mathbf{F}}$ is the vector column of the data, and the matrix $\underline{\mathbf{G}}^F$ is defined by the formula

$$\underline{\mathbf{G}}^F = \left[\iiint_{D_k}G^F_{\alpha\beta}(\mathbf{r}_j\mid\mathbf{r},\omega_l)\,E^n_\gamma(\mathbf{r},\omega_l)\,dv\right] \tag{29}$$

$$j = 1,2,\cdots M,\ k = 1,2,\cdots N,$$
$$l = 1,2,\cdots L,\ \alpha,\beta,\gamma = x,y,z.$$

In the same way we can get from (24)

$$\underline{\lambda}\underline{\mathbf{E}}^n = \underline{\mathbf{G}}^E \underline{\mathbf{m}}, \tag{30}$$

where $\underline{\lambda}$ is a block-diagonal matrix of reflectivity tensors for different subdomains D_k and different frequencies, $\underline{\mathbf{E}}^n$ is the vector column of normal field for different subdomains D_k and different frequencies, and $\underline{\mathbf{G}}^E$ is determined by expression (29) for the electric Green's tensor.

The solution of the inverse problem is reduced to the inversion of (28) with respect to $\underline{\mathbf{m}}$ and then to the inversion of (30) with respect to $\underline{\lambda}$. After that, we solve the minimization problem (26) and obtain $\Delta\widetilde{\sigma}_j$.

There are many different approaches to the solution of the linear inverse problem. We have chosen the regularized steepest descent method to invert EM data (see Appendices A and B), taking into account that the 3-D EM inversion is an ill-posed problem [*Zhdanov and Keller*, 1994]. This approach is based on the introduction of the parametric functional

$$P^\alpha(\underline{\mathbf{m}}) = \phi(\underline{\mathbf{m}}) + \alpha\, S(\underline{\mathbf{m}}), \tag{31}$$

where

$$\phi(\underline{\mathbf{m}}) = \left\| \underline{\mathbf{G}}^F \underline{\mathbf{m}} - \underline{\mathbf{F}} \right\|^2 = (\underline{\mathbf{G}}^F \underline{\mathbf{m}} - \underline{\mathbf{F}})^* (\underline{\mathbf{G}}^F \underline{\mathbf{m}} - \underline{\mathbf{F}}) \tag{32}$$

is a misfit functional and

$$S(\underline{\mathbf{m}}) = \| \underline{\mathbf{m}} - \underline{\mathbf{m}}_{\text{apr}} \|^2 = (\underline{\mathbf{m}} - \underline{\mathbf{m}}_{\text{apr}})^* (\underline{\mathbf{m}} - \underline{\mathbf{m}}_{\text{apr}}) \tag{33}$$

is a stabilizing functional. The a priori model $\underline{\mathbf{m}}_{\text{apr}}$ is some reference model, selected on the basis of all available geological and geophysical information about the area under investigation. The scalar multiplier α is a regularization parameter, and the asterisk indicates the transposed complex conjugate matrix.

The misfit functional provides the solution that best fits the observed data $\underline{\mathbf{F}}$, while the stabilizing functional ties the solution with the a priori model $\underline{\mathbf{m}}_{\text{apr}}$ and in this way keeps the solution stable.

The regularization parameter α describes the trade-off between the best fitting and reasonable stabilization. In a case when α is selected to be too small, the minimization of the parametric functional $P^\alpha(\underline{\mathbf{m}})$ is equivalent to the minimization of the misfit functional $\phi(\underline{\mathbf{m}})$, and therefore we have no regularization, which can result in unstable, incorrect solution. In a case when α is selected to be too large, the minimization of the parametric functional $P^\alpha(\underline{\mathbf{m}})$ is equivalent to the minimization of the stabilizing functional $S(\underline{\mathbf{m}})$, which will force the solution to be brought closer to the a priori model. Ultimately, we would expect the final model to be exactly like the a priori model, while the observed data are totally ignored in the inversion. Thus the critical question in the regularized solution of the inverse problem is the selection of the optimal regularization parameter α. We discuss this question in detail in Appendix B, following the basic principles of regularization theory developed by *Tikhonov and Arsenin* [1977].

Now the inversion is reduced to the solution of the minimization problem for the parametric functional:

$$P^\alpha(\underline{\mathbf{m}}) = \min. \tag{34}$$

It is shown in the regularization theory [*Tikhonov and Arsenin*, 1977; *Zhdanov and Keller*, 1994] that the solution $\underline{\mathbf{m}}^\alpha$ of the problem (34) is a continuous function of the data (so it is stable) and uniformly tends to the actual solution of the original inverse problem when $\alpha \to 0$.

We will discuss below several problems arising in the numerical realization of the inversion method.

The determination of the a priori model $\underline{\mathbf{m}}_{\text{apr}}$. We know that $\underline{\mathbf{m}}$ is equal to zero in the area outside the inhomogeneous structure, because it is proportional to the anomalous conductivity (see equation (25)). In general, if we do not know the value of $\underline{\mathbf{m}}$ and the size of the anomalous body, the best choice for $\underline{\mathbf{m}}_{\text{apr}}$ is zero function. In other words, we take the background model as the a priori model. However, if there is a priori information available (from other geological or geophysical data) about the possible model parameter $\Delta\widetilde{\sigma}$, we can use this information to determine $\underline{\lambda}_0$ and $\underline{\mathbf{m}}_0$ in the first step of the inversion. The parameter $\underline{\mathbf{m}}_0$ is used thereafter as an a priori model $\underline{\mathbf{m}}_{\text{apr}} = \underline{\mathbf{m}}_0$ in regularized inversion.

The criterion for ending the iteration for a fixed α. We use the so-called misfit condition as the criterion for ending the iteration for a fixed regularization parameter α. It is

$$\frac{\| \phi(\underline{\mathbf{m}}_n) \|}{\| \underline{\mathbf{F}} \|} \leq \delta,$$

where ϕ is a misfit functional, $\underline{\mathbf{m}}_n$ is the nth iteration in the solution of minimization problem (34) by a gradient type method (see Appendix A), and δ is a relative error (in the numerical example we set $\delta = 0.01$).

Determination of the quasi-optimum regularization parameter α. The basic principles used for determining the regularization parameter α are discussed in Appendix B. In order to avoid divergence, we begin an iteration from a large value of α (e.g., $\alpha_0 = 100.$), then reduce α ($\alpha = \alpha_0/10$) on each iteration and continuously iterate until the misfit condition is reached (see Appendix B).

Note in the conclusion of this section that in principle, the regularized solution should be applied only to the first step of the QL inversion because it can be shown that the second and the third steps involve a solution of well-posed problems.

Numerical Results

In order to test the algorithm, we have computed an EM field for two rectangular structures in a homogeneous half space (one conductive and one resistive), excited by a plane wave (Figure 1a). The observed data on the surface were simulated by forward modeling using a full integral equation code [*Xiong*, 1992]. Figure 2 shows the comparison of the full integral equation solutions (solid line) and QL approximation (dashed line) for apparent resistivities computed for the H-polarization (TM) mode (ρ_{yx}) at frequencies 10 Hz, 1 Hz, 0.5 Hz, and 0.2 Hz. Calculations are done for receivers located along the Y axes on the surface. Figures 3a and 3b present the amplitude and the phase of the apparent resistivity distribution [*Zhdanov and Keller*, 1994], calculated from the observed EM field on the surface of the Earth for frequency of 1 Hz. For 3-D EM inversion we used EM data collected along 15 profiles on the surface of the Earth for four frequencies: 10 Hz, 1 Hz, 0.5 Hz, and 0.2 Hz. In the given frequency range the displacement currents are negligibly smaller than the conductive currents, so the inversion was applied only for the conductivity distribution. In the numerical test we have selected 144 substructures for inversion, shown in Figure 1b, and we have used the additional simplification that the reflectivity tensor $\widehat{\lambda}$ is scalar and constant within every substructure. The results of inversion for different data (with and without 5% random noise) are shown in the following figures.

Figure 4 presents a vertical slice along the line $x = 0$ of the results of inversion of the noise free data. One can clearly see the cross sections of conductive and resistive bodies on this picture, however, the upper parts of the anomalous bodies are resolved slightly better than the lower parts, as can be explained by the fact that EM field is less sensitive to the lower parts of anomalous structures. Figure 5 shows the vertical slice along the line $x = 300$ m that passes outside the bodies with anomalous conductivities. We can see now only background conductivity on this cross section with a very weak variation that corresponds very well to the original model. Figure 6 presents the volume image of the inverted model. The result clearly shows the anomalous bodies with the low and high resistivities. The relationship between the misfit functional and the number of iterations is shown in Figure 7. Figure 8 presents the results of the inversion of the same data, but with

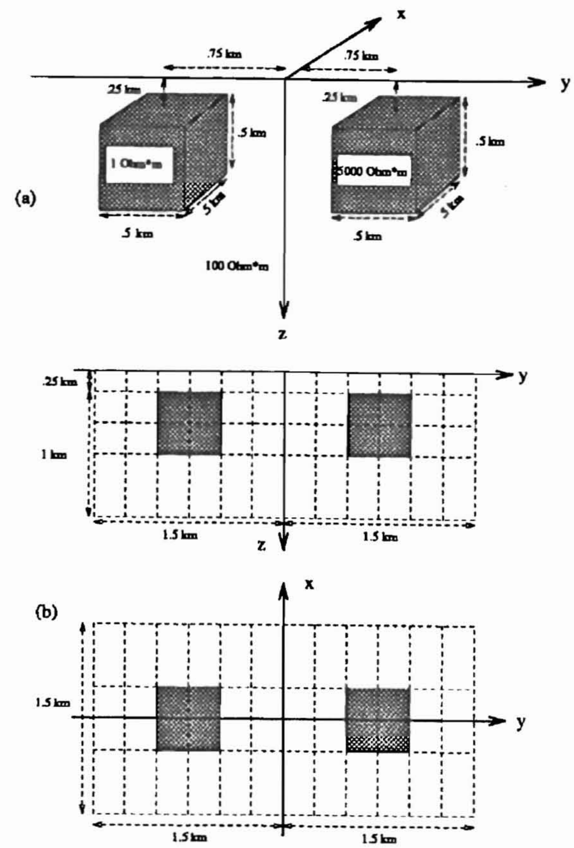

Figure 1. (a) Three-dimensional model of rectangular conductive and resistive structures in a homogeneous half space, excited by the plane wave (b) Scheme of the division of the model on substructures used for inversion.

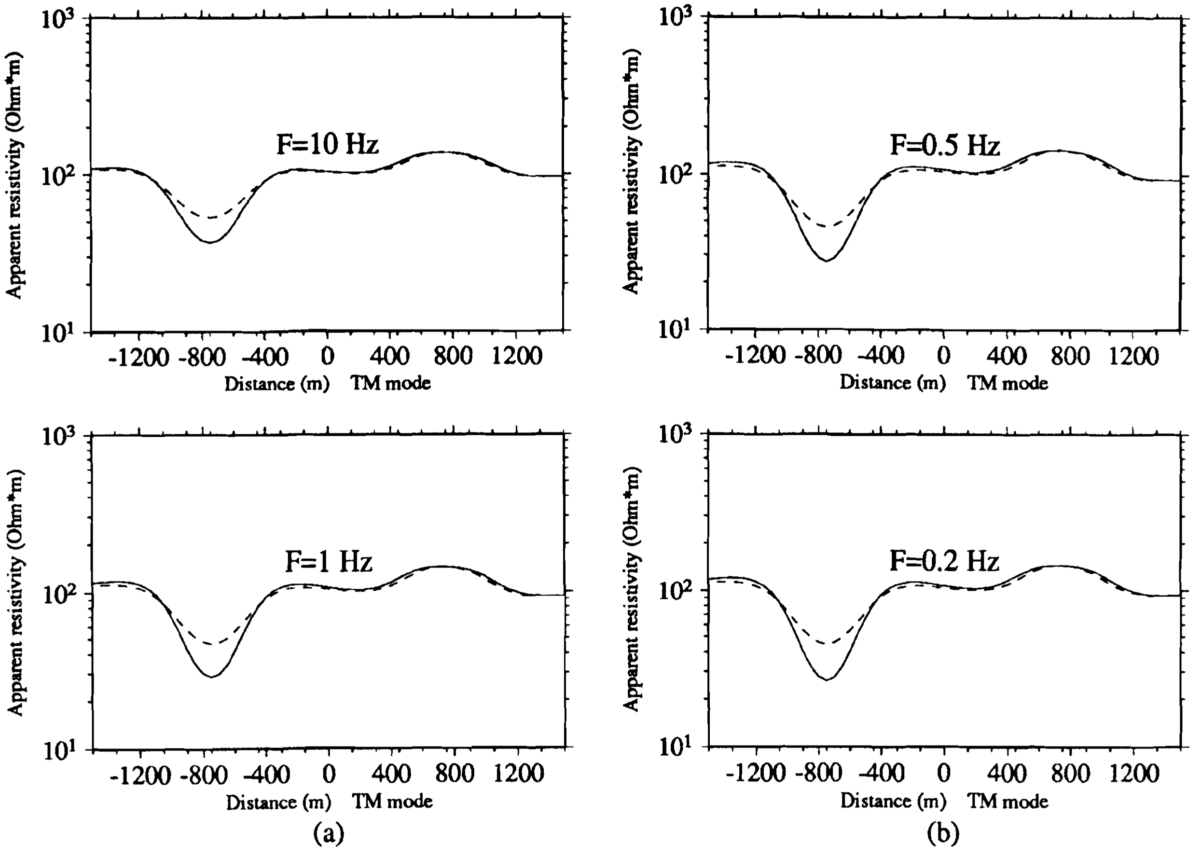

Figure 2. Numerical comparison of the full integral equation solution (solid line) and the QL approximation (dashed line) computed for the model shown in Figure 1 at the frequencies 10 Hz, 1 Hz, 0.5Hz, and 0.2 Hz. Calculations are done for the receivers, located along the axes Y on the surface. Plots present apparent resistivities computed for the TM mode (ρ_{yx}).

5% noise added to the real and imaginary parts of the apparent resistivity. One can see that the noise practically did not affect the result. This can be explained simply by using a regularized solution.

In inverting these models, we have used 5400 cells for forward modeling. For such a large number of cells the forward modeling using a full integral equation code [*Xiong*, 1992; *Xiong and Kirsh*, 1992] takes several hours on a SPARC-10 workstation for one iteration. In this example we used about 50 iterations in the inversion scheme, so it would take several days on a SPARC-10 workstation to solve this inverse problem. However, the quasi-linear inversion takes only about 1 hour.

These results demonstrate that the quasi-linear inverse algorithm is a fast and powerful tool for 3-D EM inversion.

Conclusion

We have developed a rapid 3-D electromagnetic inversion algorithm based on the QL approximation of the forward modeling. The method can be applied for models with various sources of excitation, including plane waves for magnetotellurics, horizontal bipoles, vertical bipoles, horizontal rectangular loops, vertical magnetic dipoles, and the loop-loop system for airborne electromagnetics.

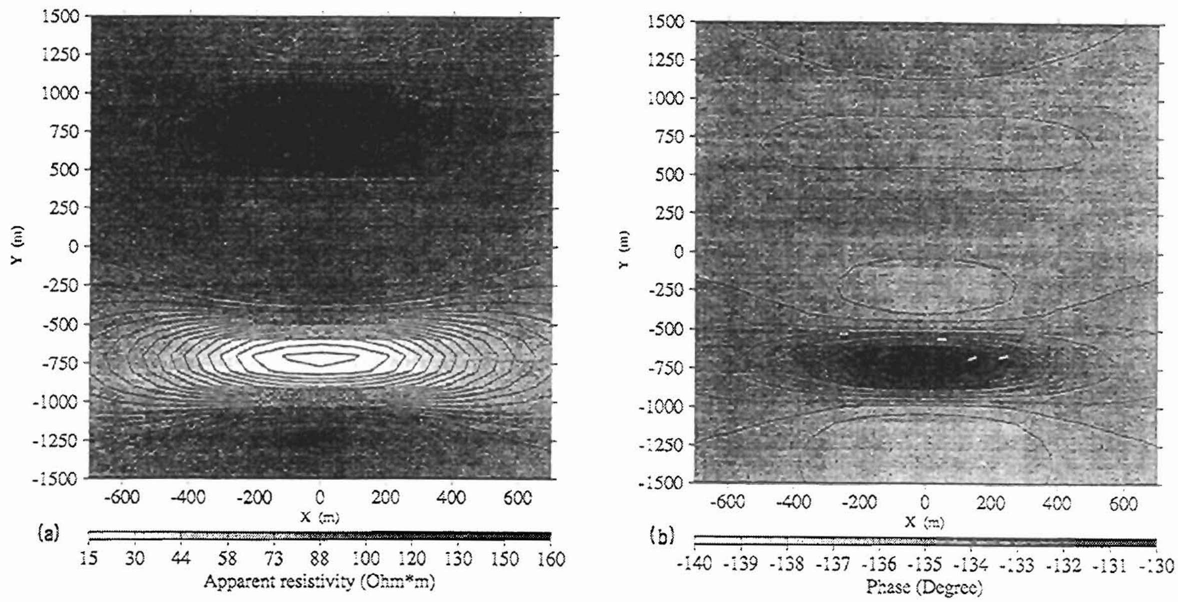

Figure 3. (a) Apparent resistivity amplitude distribution for the model shown in Figure 1, calculated on the surface of the Earth for frequency of 1 Hz. (b) Impedance phase distribution for the model shown in Figure 1, calculated for frequency of 1 Hz.

The main advantage of the method is that we reduce the original nonlinear inverse problem to a set of linear inverse problems to obtain a rapid 3-D conductivity inversion. The QL inverse problem is solved by a regularized gradient type method which ensures stability and rapid convergence. Synthetic examples (with random noise or without) of inversion demonstrate that the algorithm for inverting 3-D EM data is fast and stable.

Appendix A: Regularized Steepest Descent Method for Solving Linear Inverse Problem

In order to obtain a stable solution of equations (28) and (30), we introduced the parametric functional

$$P^{\alpha}(\underline{\mathbf{m}}) = \phi(\underline{\mathbf{m}}) + \alpha\, S(\underline{\mathbf{m}}),$$

where functionals $\phi(\underline{\mathbf{m}})$ and $S(\underline{\mathbf{m}})$ were determined by equations (32) and (33).

To solve the minimization problem (34), we calculate the first variation of the parametric functional:

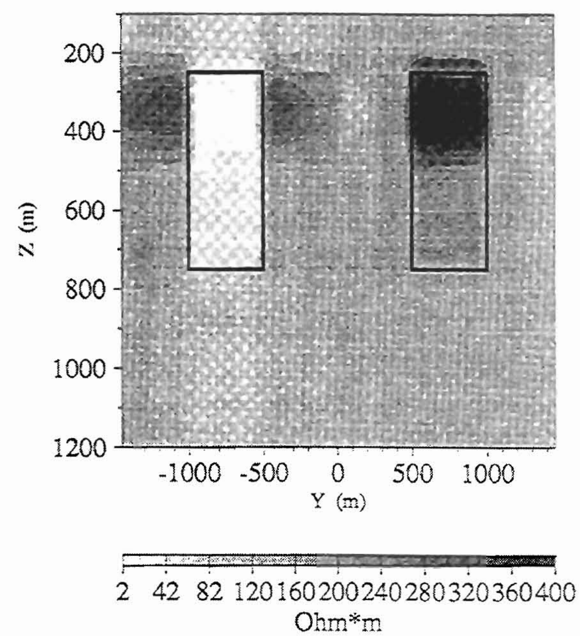

Figure 4. The vertical slice along the line $x = 0$ of the results of inversion of the noise free data for the model shown in Figure 1.

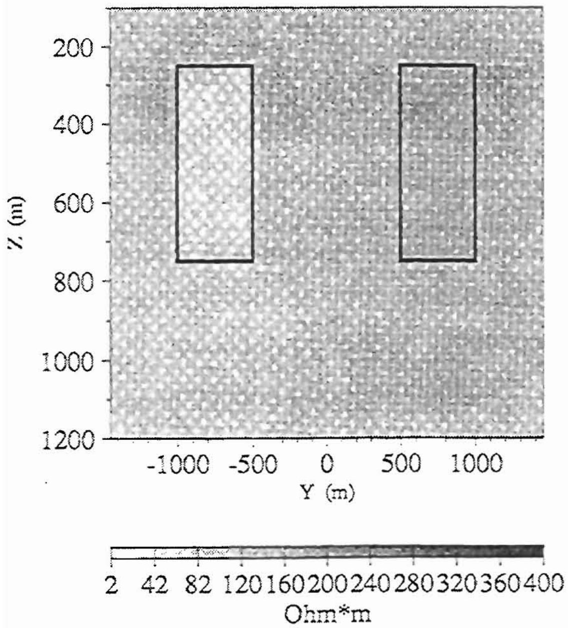

Figure 5. The vertical slice along the line $x = 300$ m of the results of inversion of the noise free data for the model shown in Figure 1.

$$\delta P^{\alpha}(\underline{\mathbf{m}}) =$$
$$2\,\mathrm{Re}\,\{\delta \underline{\mathbf{m}}^{*}\,[\,\underline{\mathbf{G}}^{F*}(\underline{\mathbf{G}}^{F}\,\underline{\mathbf{m}} - \underline{\mathbf{F}}) + \alpha\,(\underline{\mathbf{m}} - \underline{\mathbf{m}}_{\mathrm{apr}})\,]\,\}.$$

Let us select $\delta\underline{\mathbf{m}}$ as

$$\delta \underline{\mathbf{m}} = -k^{\alpha}\,\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}})\ , \quad 0 < k^{\alpha} < \infty, \tag{A1}$$

where

$$\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}}) = \underline{\mathbf{G}}^{F*}(\underline{\mathbf{G}}^{F}\,\underline{\mathbf{m}} - \underline{\mathbf{F}}) + \alpha\,(\underline{\mathbf{m}} - \underline{\mathbf{m}}_{\mathrm{apr}}). \tag{A2}$$

This selection makes

$$\delta P^{\alpha}(\underline{\mathbf{m}}) = -2\,k^{\alpha}\mathrm{Re}\{\underline{\mathbf{l}}^{\alpha*}(\underline{\mathbf{m}})\,\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}})\} < 0.$$

That means that the parametric functional is reduced if we apply perturbation (A1) to the model parameters.

We construct an iteration process as follows:

$$\underline{\mathbf{m}}^{\alpha}_{n+1} = \underline{\mathbf{m}}^{\alpha}_{n} + \delta\underline{\mathbf{m}} = \underline{\mathbf{m}}^{\alpha}_{n} - k^{\alpha}_{n}\,\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}}_{n}). \tag{A3}$$

The coefficient k^{α}_{n} can be determined from the condition

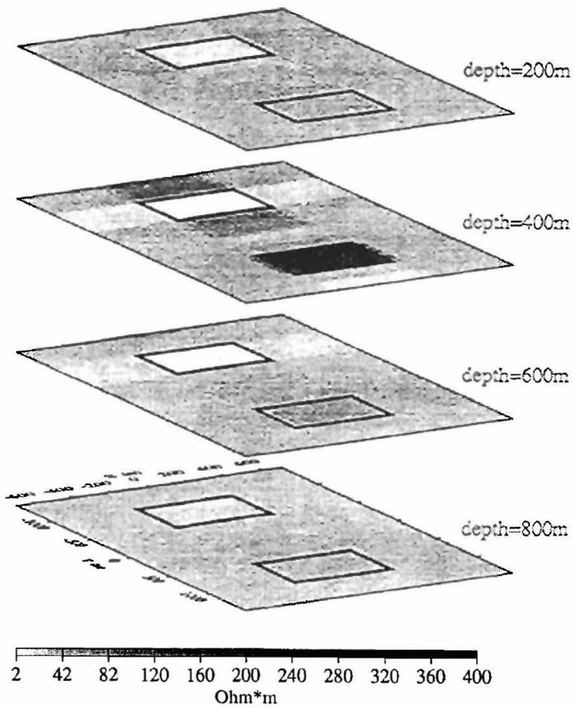

Figure 6. The volume image of the inverted model computed from EM data collected along 15 profiles on the surface of the Earth for frequencies of 10 Hz, 1 Hz, 0.5 Hz, and 0.2 Hz, for the model shown in Figure 1.

$$P^{\alpha}(\underline{\mathbf{m}}^{\alpha}_{n+1}) = P^{\alpha}(\underline{\mathbf{m}}^{\alpha}_{n} - k^{\alpha}_{n}\,\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}}_{n})) = f(k^{\alpha}_{n}) = \min.$$

Solution of this minimization problem gives the following best estimation for the length of the step:

$$k^{\alpha}_{n} = \frac{\underline{\mathbf{l}}^{\alpha*}(\underline{\mathbf{m}}^{\alpha}_{n})\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}}^{\alpha}_{n})}{\underline{\mathbf{l}}^{\alpha*}(\underline{\mathbf{m}}^{\alpha}_{n})(\underline{\mathbf{G}}^{F*}\underline{\mathbf{G}}^{F} + \alpha\underline{\mathbf{I}})\underline{\mathbf{l}}^{\alpha}(\underline{\mathbf{m}}^{\alpha}_{n})}. \tag{A4}$$

According to equations (A2), (A3) and (A4), we can obtain $\underline{\mathbf{m}}$ by iterations.

Appendix B: Determination of the Quasi-Optimum Regularization Parameter

The key component of the regularizing algorithm construction is the determination of the optimum value of the regularization parameter. This problem

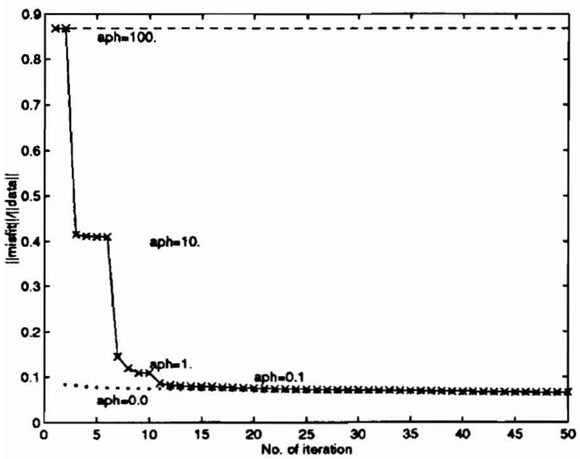

Figure 7. The plot of the misfit functional as a function of the number of iterations, calculated during inversion of the data for the model shown in Figure 1.

can be solved on the basis of a priori information about the accuracy of the observed data.

Suppose that the data $\underline{\mathbf{F}}_\delta$ were observed with some noise $\underline{\mathbf{F}}_\delta = \underline{\mathbf{F}}_t + \delta\underline{\mathbf{F}}$, where $\underline{\mathbf{F}}_t$ is the true solution of the problem and the level of the errors in the observed data is equal to δ:

$$\|\underline{\mathbf{F}}_\delta - \underline{\mathbf{F}}_t\| \leq \delta. \tag{B1}$$

Then the regularization parameter can be determined by the misfit condition

$$\| \hat{G}_d \underline{\mathbf{m}}^\alpha - \underline{\mathbf{F}} \|^2 = \delta, \tag{B2}$$

where $\underline{\mathbf{m}}^\alpha$ is the solution of minimization problem (34) for the given value of α.

We introduce the following notations:

$$p(\alpha) = P^\alpha(\underline{\mathbf{m}}^\alpha);$$

$$i(\alpha) = \| \underline{\mathbf{G}}^F \underline{\mathbf{m}}^\alpha - \underline{\mathbf{F}} \|^2;$$

$$s(\alpha) = S(\underline{\mathbf{m}}^\alpha).$$

Let us examine some properties of the functions $p(\alpha), i(\alpha), s(\alpha)$. First of all, it is known that functions $p(\alpha), i(\alpha)$, and $s(\alpha)$ are monotone functions: $p(\alpha)$ and $i(\alpha)$ are nondecreasing, and $s(\alpha)$ is nonincreasing [*Tikhonov and Arsenin*, 1977; *Zhdanov and Keller*, 1994]. It is important to notice that the functions $p(\alpha), i(\alpha), s(\alpha)$ can be proved to be continuous

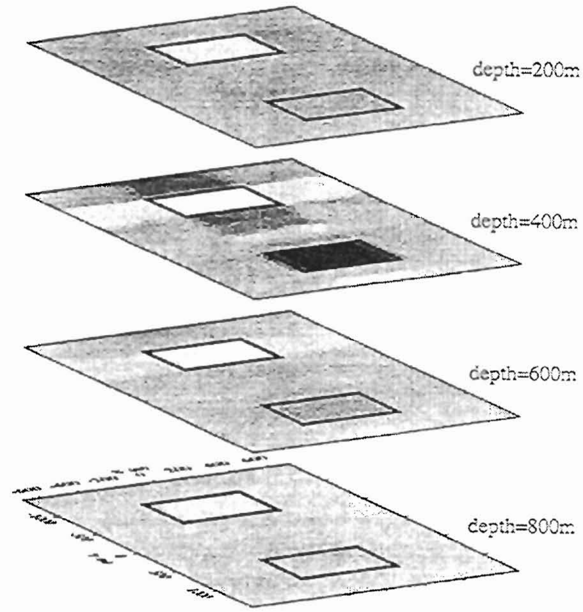

Figure 8. The results of the inversion of the same EM data (shown in Figure 3) collected along 15 profiles on the surface of the Earth for frequencies of 10 Hz, 1 Hz, 0.5 Hz, and 0.2 Hz, for the model shown in Figure 1, but with 5% noise added.

functions (if the element $\underline{\mathbf{m}}^\alpha$ is unique). Note also that

$$p(\alpha) \to 0 \text{ for } \alpha \to 0,$$

and

$$p(0) = 0, \; i(0) = 0. \tag{B3}$$

Thus we have the following result: if $i(\alpha)$ is a one-to-one function, then for any positive number $\delta < \delta_0 = \| \underline{\mathbf{G}}^F \underline{\mathbf{m}}_{\mathrm{apr}} - \underline{\mathbf{F}} \|^2$ (where $\underline{\mathbf{m}}_{\mathrm{apr}}$ is some a priori model) there exists $\alpha(\delta)$ such that

$$\| \underline{\mathbf{G}}^F \underline{\mathbf{m}}^{\alpha(\delta)} - \underline{\mathbf{F}} \|^2 = \delta. \tag{B4}$$

Note that $i(\alpha)$ is a one-to-one function when the element $\underline{\mathbf{m}}_\alpha$ is unique. It happens, for example, in the case under consideration when we have a linear inverse problem , and $s(\underline{\mathbf{m}})$ is a quadratic functional [*Tikhonov and Arsenin*, 1977].

Let us consider one simple numerical method to determine the parameter α. Consider for example the progression of numbers

$$\alpha_k = \alpha_0 q^k; \; k = 0, 1, 2, ..., n; \; q > 0. \tag{B5}$$

For any number α_k we can find an element $\underline{\mathbf{m}}_{\alpha_k}$, minimizing $P^{\alpha_k}(\underline{\mathbf{m}})$, and calculate the misfit $\| \underline{\mathbf{G}}^F \underline{\mathbf{m}}^{\alpha_k} - \underline{\mathbf{F}} \|^2$. The optimal value of the parameter α is the number α_{k0}, for which we have

$$\| \underline{\mathbf{G}}^F \underline{\mathbf{m}}^{\alpha_{k0}} - \underline{\mathbf{F}} \|^2 = \delta. \tag{B6}$$

Equality (B6) is called the "misfit condition".

Note, in conclusion, that introducing the misfit functional and misfit condition in the solution of the inverse problem is the direct way to use a priori geologic and geophysical information about the Earth's structure and the quality of the observed data to reduce the ambiguity and increase the stability of the solution.

Appendix C: Calculation of the Anomalous Conductivity

The anomalous conductivity $\Delta\widetilde{\sigma}_j$can be obtained by solving the minimization problem (26).

Let us calculate the first variation of the functional $\varphi(\Delta\widetilde{\sigma}_j)$:

$$\delta\varphi(\Delta\widetilde{\sigma}_j, \delta\Delta\widetilde{\sigma}_j) =$$

$$-2\mathrm{Re}\sum_{j=1}^{N}\sum_{l=1}^{L}\delta\Delta\widetilde{\sigma}_j(\omega_l)\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* \cdot\left[m^j_{\beta\gamma}(\omega_l)-\Delta\widetilde{\sigma}_j(\omega_l)\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)\right]. \tag{C1}$$

The minimum of (26) is reached when $\delta\varphi(\Delta\widetilde{\sigma}_j, \delta\Delta\widetilde{\sigma}_j) \equiv 0$ for any $\delta\Delta\widetilde{\sigma}_j$. Therefore

$$\Delta\widetilde{\sigma}_j(\omega_l)\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* \cdot\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right) = \sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* m^j_{\beta\gamma}, \tag{C2}$$

where $\beta,\gamma = x,y,z, \quad j = 1,2,\cdots,N$.

Thus we have

$$\Delta\widetilde{\sigma}_j(\omega_l) = \sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* m^j_{\beta\gamma} \cdot\left\{\sum_{\beta\gamma=x,y,z}\left[\left(\delta_{\beta\gamma}+Re\lambda^j_{\beta\gamma}(\omega_l)\right)^2 + \left(Im\lambda^j_{\beta\gamma}(\omega_l)\right)^2\right]\right\}. \tag{C3}$$

In practical applications anomalous electrical conductivity $\Delta\sigma$ and anomalous dielectric permittivity $\Delta\varepsilon$ usually do not depend on frequency ω. In this case $\Delta\widetilde{\sigma}_j(\omega_l) = \Delta\sigma_j - i\omega_l\Delta\varepsilon_j$, and condition (C2) should be rewritten as

$$\Delta\sigma_j\sum_{l=1}^{L}\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* \cdot\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right) - i\Delta\varepsilon_j\sum_{l=1}^{L}\omega_l\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* \cdot\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right) = \sum_{l=1}^{L}\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* m^j_{\beta\gamma} \tag{C4}$$

From (C4) we find immediately the expressions for anomalous conductivity

$$\Delta\sigma_j = \mathrm{Re}\sum_{l=1}^{L}\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* m^j_{\beta\gamma} \cdot\left\{\sum_{l=1}^{L}\sum_{\beta\gamma=x,y,z}\left[\left(\delta_{\beta\gamma}+Re\lambda^j_{\beta\gamma}(\omega_l)\right)^2 + \left(\mathrm{Im}\lambda^j_{\beta\gamma}(\omega_l)\right)^2\right]\right\}^{-1}. \tag{C5}$$

and for anomalous dielectric permittivity

$$\Delta\varepsilon_j = -\mathrm{Im}\sum_{l=1}^{L}\sum_{\beta\gamma=x,y,z}\left(\delta_{\beta\gamma}+\lambda^j_{\beta\gamma}(\omega_l)\right)^* m^j_{\beta\gamma} \cdot\left\{\sum_{l=1}^{L}\omega_l\sum_{\beta\gamma=x,y,z}\left[\left(\delta_{\beta\gamma}+\mathrm{Re}\lambda^j_{\beta\gamma}(\omega_l)\right)^2 + \left(\mathrm{Im}\lambda^j_{\beta\gamma}(\omega_l)\right)^2\right]\right\}^{-1}. \tag{C6}$$

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References

Berdichevsky, M. N., and M. S. Zhdanov, *Advanced Theory of Deep Geomagnetic Sounding*, Elsevier, New York, 1984.

Born, M., *Optic*, Springer-Verlag, New York, 1933.

Druskin, V., and L. Knizhnerman, Spectral approach to solving three-dimensional Maxwell's diffusion equations in the time and frequency domains, *Radio Sci., 29(4)*, 937-953, 1994.

Eaton, P. A., 3-D electromagnetic inversion using integral equations, *Geophys. Prospect, 37*, 407-426, 1989.

Habashy, T. M., R. W. Groom, and B. R. Spies, Beyond the Born and Rytov approximations: A nonlinear approach to electromagnetic scattering, *J. Geophys. Res., 98 (B2)*, 1759-1775, 1993.

Hohmann, G. W. , Three-dimensional induced polarization and EM modeling, *Geophysics, 40*, 309-324, 1975.

Jupp, D. L. B., and K. Vozoff, Two-dimensional magnetotelluric inversion, *Geophys. J. R. Astron. Soc., 50*, 333-352, 1977.

Madden, T. R., and R. L. Mackie, Three-dimensional magnetotelluric modelling and inversion, *Proc. IEEE, 77(2)*, 318-332, 1989.

Newman, G. A., and D. L. Alumbaugh, 3-D massively parallel electromagnetic inversion, *in PIERS Proceedings, p.324, Univ. of Wash., Seattle*, 1995.

Oristaglio, M. L., and M. H. Worthington, Inversion of surface and borehole EM data for two-dimensional electrical conductivity models, *Geophys. Prospect, 28*, 633-657, 1980.

Smith, J .T., and J. R. Booker, Rapid inversion of two- and three-dimensional magnetotelluric data, *J. Geophys. Res., 96 (B3)*, 3905-3922, 1991.

Tikhonov, A. N., and V. Y. Arsenin, *Solution of Ill-Posed Problems*, Wiley, New York, 1977.

Torres-Verdin, C. and T. M. Habashy, Rapid 2.5-dimensional forward modeling and inversion via a new nonlinear scattering approximation, *Radio Sci., 29 (4)*, 1051-1079, 1994.

Torres-Verdin, C., and T. M. Habashy, Two-steps linear inversion, *IEEE Trans. Antennas Propagat., 43 (4)*, 405-415, 1995a.

Torres-Verdin, C., and T. M. Habashy, An overview of the extended Born approximation as a nonlinear scattering approach, and its application to cross-well resistivity imaging, *in PIERS Proceedings, p.321, Univ. of Wash., Seattle*, 1995b.

Tripp, A. C., Group theoretic reduction of the EM impedance matrix for large-contrast symmetric prisms in a layered Earth, *Pure Applied Geophys., 133*, 127-147, 1990.

Tripp, A. C., and G. W. Hohmann, Three-dimensional EM cross-well inversion, *IEEE Trans. Geosci. Remote Sens., 31*, 121-126, 1993.

Wannamaker, P. E., Advances in 3-D magnetotelluric modeling using integral equations, *Geophysics, 56*, 1716-1728, 1991.

Weidelt, P., EM induction in three-dimensional structures, *J. Geophys., 41*, 85-109, 1975.

Xie, G., and K.H. Lee, Nonlinear inversion of 3-D electromagnetic data, *in PIERS Proceedings, p.323, Univ. of Wash., Seattle*, 1995.

Xiong, Z., EM modeling of three-dimensional structures by the method of system iteration using integral equations, *Geophysics, 57*, 1556-1561, 1992.

Xiong, Z. and A. Kirsch, Three-dimensional earth conductivity inversion, *J. Comput. Appl. Math., 42* , 109-121, 1992.

Xiong Z., and A. C. Tripp, Scattering matrix evaluation using spatial symmetry in EM modeling, *Geophys. J. Int., 114*, 459-464, 1993.

Zhdanov, M. S., Regularization in inversion theory, tutorial, Colo. Sch. of Mines, Golden, 1993.

Zhdanov, M. S, and S. Fang, Quasi- linear approximation in 3-D EM modeling, *Geophysics, 61 (3)*, 1996.

Zhdanov, M. S., and N. G. Golubev, Use of the finite functions method for the solution of the 2D inverse problem, *J. Geomagn. Geoelectr., 35*, 707-722, 1983.

Zhdanov, M. S., and G. Keller, *The Geoelectrical Methods in Geophysical Exploration*, 900 pp., Elsevier, New York, 1994.

Zhdanov, M. S., and I. M. Varentsov, Interpretation of local two-dimensional EM anomalies by formalized trial procedures, *Geophys. J. R. Astron. Soc., 75*, 623-638, 1983.

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