Chapter 5

NONLINEAR APPROXIMATIONS FOR ELECTROMAGNETIC SCATTERING FROM ELECTRICAL AND MAGNETIC INHOMOGENEITIES

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Abstract: We extend linear and nonlinear approximations for electromagnetic fields to a medium with inhomogeneous distribution of both electrical and magnetic material properties. These approximations are presented in the form of tensor integrals over a domain with anomalous parameters. The developed approximations combine the linear and nonlinear estimations depending on the ratio of complex conductivity and magnetic susceptibility perturbations. These approximations form a basis for fast EM modeling and imaging in multi-dimensional environments where joint electrical and magnetic inhomogeneity is an essential feature of the model. Numerical tests carried out for one-dimensional electromagnetic logging applications demonstrate the validity of the theory and the effectiveness of the proposed approximations.

1. INTRODUCTION

Traditionally, in modeling electromagnetic fields in geophysical explorations, one takes into account the distribution of the anomalous electrical conductivity only. However, there are many practical situations when a conductive object has significant magnetic properties as well. For example, a magnetite-containing ore body, some geological formations of sedimentary or volcanic origin, and drilling mud with heavy material ingredients are characterized by both anomalous conductivity and magnetic susceptibility, which can produce significant effects on the electromagnetic tool response.

The foundations of the integral equation method were developed in pioneering works by Hohmann (1975), Tabarovsky (1975), Weidelt (1975), etc. Recently developed localized nonlinear (Habashy et al., 1993; Torres-Verdin and Habashy, 1994), quasi-linear (Zhdanov and Fang, 1996, 1997) and quasi-analytical (Zhdanov et al., 2000) approximations in an electrically inhomogeneous medium are the basis for high-performance forward modeling and inversion methods. Some theoretical aspects of Born type approximations for magnetic properties were considered by Murray et al. (1999) and Cheryauka and Sato (1999). In this paper we extend these approaches and formulate linear and nonlinear approximations for a model with joint fluctuations of electrical and magnetic properties.

Synthetic modeling examples illustrating the comparison with analytical solutions for one-dimensional induction logging and casing-scanning problems show the areas of potential applications of these nonlinear algorithms.

2. INTEGRAL EQUATION FORMULATION

Let us consider the general 3-D EM forward problem illustrated in Figure 1. A medium with joint electrical and magnetic inhomogeneities is excited by electrical, $\mathbf{J}^{\text{inc}}$, and magnetic, $\mathbf{M}^{\text{inc}}$, harmonic currents distributed within a domain V_{inc}. Time dependence is $e^{-i\omega t}$.

A lossy unbounded medium is characterized by a complex electrical conductivity $\tilde{\sigma}(\mathbf{r}) = \sigma(\mathbf{r}) - i\omega\varepsilon(\mathbf{r})$, and a magnetic susceptibility $\chi(\mathbf{r})$, where $\sigma(\mathbf{r})$ and $\varepsilon(\mathbf{r})$ are electrical conductivity and dielectric constant respectively. To symmetrize further considerations, we introduce a complex magnetic permeability $\tilde{\mu}$:

$$\tilde{\mu}(\mathbf{r}) = i\omega\mu_0(1 + \chi(\mathbf{r})),$$

where μ_0 is the free space magnetic permeability. Note that, in general cases, $\tilde{\sigma}(\mathbf{r})$, $\tilde{\mu}(\mathbf{r})$ can be frequency-dependent and, in anisotropic media, can be represented by 3×3 dyadic functions. In this paper we will study the isotropic medium only.

We assume that the material property distributions $\tilde{\sigma}(\mathbf{r})$, $\tilde{\mu}(\mathbf{r})$ are expressed by a sum of background, $\tilde{\sigma}_b(\mathbf{r})$, $\tilde{\mu}_b(\mathbf{r})$, and anomalous, $\tilde{\sigma}_a(\mathbf{r})$, $\tilde{\mu}_a(\mathbf{r})$, distributions:

$$\begin{aligned}\tilde{\sigma}(\mathbf{r}) &= \tilde{\sigma}_b(\mathbf{r}) + \tilde{\sigma}_a(\mathbf{r}),\\ \tilde{\mu}(\mathbf{r}) &= \tilde{\mu}_b(\mathbf{r}) + \tilde{\mu}_a(\mathbf{r}).\end{aligned} \tag{5.1}$$

We assume that anomalous distributions are nonzero only within the corresponding domains V_σ and V_μ. The electromagnetic fields **E**, **H** in this model can be presented as a

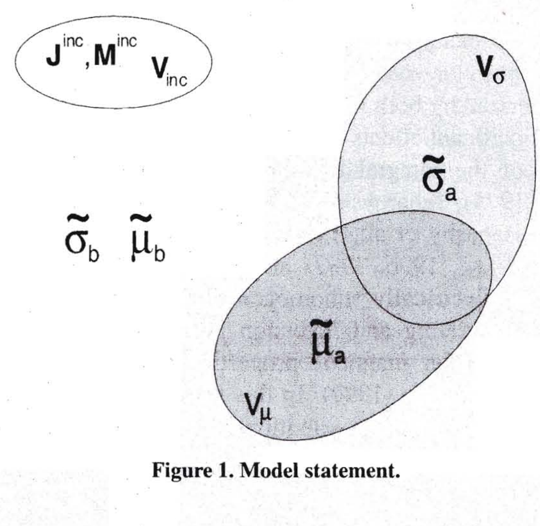

Figure 1. Model statement.

superposition of background $\mathbf{E}^{\mathrm{b}}$, $\mathbf{H}^{\mathrm{b}}$ and anomalous $\mathbf{E}^{\mathrm{a}}$, $\mathbf{H}^{\mathrm{a}}$ fields,

$$\begin{aligned}
&\mathbf{E} = \mathbf{E}^{\mathrm{b}} + \mathbf{E}^{\mathrm{a}}, \\
&\mathbf{H} = \mathbf{H}^{\mathrm{b}} + \mathbf{H}^{\mathrm{a}},
\end{aligned} \tag{5.2}$$

which satisfy Maxwell equations

$$\begin{aligned}
&\nabla \times \mathbf{H}^{\mathrm{b}} - \tilde{\sigma}_{\mathrm{b}}(\mathbf{r})\mathbf{E}^{\mathrm{b}} = \mathbf{J}^{\mathrm{inc}}, \\
&\nabla \times \mathbf{E}^{\mathrm{b}} - \tilde{\mu}_{\mathrm{b}}(\mathbf{r})\mathbf{H}^{\mathrm{b}} = \mathbf{M}^{\mathrm{inc}},
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\nabla \times \mathbf{H}^{\mathrm{a}} - \tilde{\sigma}_{\mathrm{b}}(\mathbf{r})\mathbf{E}^{\mathrm{a}} = \tilde{\sigma}_{\mathrm{a}}(\mathbf{r})(\mathbf{E}^{\mathrm{b}} + \mathbf{E}^{\mathrm{a}}), \\
&\nabla \times \mathbf{E}^{\mathrm{a}} - \tilde{\mu}_{\mathrm{b}}(\mathbf{r})\mathbf{H}^{\mathrm{a}} = \tilde{\mu}_{\mathrm{a}}(\mathbf{r})(\mathbf{H}^{\mathrm{b}} + \mathbf{H}^{\mathrm{a}}).
\end{aligned} \tag{5.4}$$

The background field can be derived using the Green's function method (Felsen and Marcuvitz, 1994):

$$\begin{aligned}
&\mathbf{E}^{\mathrm{b}} = \left\langle \hat{\mathbf{G}}^{JE}\mathbf{J}^{\mathrm{inc}} + \hat{\mathbf{G}}^{ME}\mathbf{M}^{\mathrm{inc}} \right\rangle_{V_{\mathrm{inc}}}, \\
&\mathbf{H}^{\mathrm{b}} = \left\langle \hat{\mathbf{G}}^{JH}\mathbf{J}^{\mathrm{inc}} + \hat{\mathbf{G}}^{MH}\mathbf{M}^{\mathrm{inc}} \right\rangle_{V_{\mathrm{inc}}},
\end{aligned} \tag{5.5}$$

where we use the notation

$$\left\langle \hat{\mathbf{G}}\mathbf{J} \right\rangle_V = \int_V \hat{\mathbf{G}}(\mathbf{r}|\mathbf{r}')\mathbf{J}(\mathbf{r}')\mathrm{d}\mathbf{r}'.$$

In the last formula, $\hat{\mathbf{G}}^{JE}, \hat{\mathbf{G}}^{JH}, \hat{\mathbf{G}}^{MH}$, and $\hat{\mathbf{G}}^{ME}$ are tensor Green's functions satisfying the following second-order differential equations:

$$\begin{aligned}
&\nabla \times \frac{1}{\tilde{\mu}_{\mathrm{b}}(\mathbf{r})} \nabla \times \hat{\mathbf{G}}^{JE}(\mathbf{r}'|\mathbf{r}) - \tilde{\sigma}_{\mathrm{b}}(\mathbf{r})\hat{\mathbf{G}}^{JE}(\mathbf{r}'|\mathbf{r}) = \hat{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}'), \\
&\hat{\mathbf{G}}^{JH}(\mathbf{r}'|\mathbf{r}) = \frac{1}{\tilde{\mu}_{\mathrm{b}}(\mathbf{r})} \nabla \times \hat{\mathbf{G}}^{JE}(\mathbf{r}'|\mathbf{r}), \\
&\nabla \times \frac{1}{\tilde{\sigma}_{\mathrm{b}}(\mathbf{r})} \nabla \times \hat{\mathbf{G}}^{MH}(\mathbf{r}'|\mathbf{r}) - \tilde{\mu}_{\mathrm{b}}(\mathbf{r})\hat{\mathbf{G}}^{MH}(\mathbf{r}'|\mathbf{r}) = \hat{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}'), \\
&\hat{\mathbf{G}}^{ME}(\mathbf{r}'|\mathbf{r}) = \frac{1}{\tilde{\sigma}_{\mathrm{b}}(\mathbf{r})} \nabla \times \hat{\mathbf{G}}^{MH}(\mathbf{r}'|\mathbf{r}),
\end{aligned}$$

where $\hat{\mathbf{I}}$ is the identity tensor, and $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac function (Tai, 1979).

The volume densities of electrical and magnetic anomalous currents $\mathbf{J}^{\mathrm{a}}$, $\mathbf{M}^{\mathrm{a}}$ are equal to

$$\begin{aligned}
&\mathbf{J}^{\mathrm{a}}(\mathbf{r}) = \tilde{\sigma}_{\mathrm{a}}(\mathbf{r})(\mathbf{E}^{\mathrm{b}} + \mathbf{E}^{\mathrm{a}}), \\
&\mathbf{M}^{\mathrm{a}}(\mathbf{r}) = \tilde{\mu}_{\mathrm{a}}(\mathbf{r})(\mathbf{H}^{\mathrm{b}} + \mathbf{H}^{\mathrm{a}}).
\end{aligned} \tag{5.6}$$

Therefore, the anomalous field excited by arbitrary anomalous currents $\mathbf{J}^{\mathrm{a}}$, $\mathbf{M}^{\mathrm{a}}$ can be expressed by a formula similar to Equation (5.5):

$$\begin{aligned}
&\mathbf{E}^{\mathrm{a}} = \left\langle \hat{\mathbf{G}}^{JE}\mathbf{J}^{\mathrm{a}} \right\rangle_{V_\sigma} + \left\langle \hat{\mathbf{G}}^{ME}\mathbf{M}^{\mathrm{a}} \right\rangle_{V_\mu}, \\
&\mathbf{H}^{\mathrm{a}} = \left\langle \hat{\mathbf{G}}^{JH}\mathbf{J}^{\mathrm{a}} \right\rangle_{V_\sigma} + \left\langle \hat{\mathbf{G}}^{MH}\mathbf{M}^{\mathrm{a}} \right\rangle_{V_\mu}.
\end{aligned} \tag{5.7}$$

3. BORN APPROXIMATION

The conventional Born approximation, $\mathbf{E}^B$, $\mathbf{H}^B$, is based on an assumption that within domains V_σ and V_μ the anomalous fields $\mathbf{E}^a$, $\mathbf{H}^a$ are negligibly small in comparison with the background fields $\mathbf{E}^b$, $\mathbf{H}^b$:

$$\begin{array}{l}\mathbf{E}^a(\mathbf{r}) \approx 0,\\ \mathbf{H}^a(\mathbf{r}) \approx 0,\end{array} \quad \mathbf{r} \in V_\sigma \cup V_\mu. \tag{5.8}$$

In this case, according to Equation (5.6), Formula (5.7) for anomalous fields outside domains V_σ and V_μ is simplified:

$$\begin{aligned}\mathbf{E}^a &\approx \mathbf{E}^B = \langle \hat{\mathbf{G}}^{JE}\mathbf{J}^B\rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{ME}\mathbf{M}^B\rangle_{V_\mu},\\ \mathbf{H}^a &\approx \mathbf{H}^B = \langle \hat{\mathbf{G}}^{JH}\mathbf{J}^B\rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{MH}\mathbf{M}^B\rangle_{V_\mu},\end{aligned} \tag{5.9}$$

where Born current densities $\mathbf{J}^B$, $\mathbf{M}^B$ are

$$\mathbf{J}^B = \tilde{\sigma}_a \mathbf{E}^b, \qquad \mathbf{M}^B = \tilde{\mu}_a \mathbf{H}^b.$$

In fact, this approximation is linear with respect to the anomalous material properties $\tilde{\sigma}_a$, $\tilde{\mu}_a$ and can be treated as a first-order term in a complete Born–Neumann series. These series of limited numbers of terms can be treated as nonlinear approximations; however, their convergence, in most cases, is problematical.

4. LOCALIZED APPROXIMATION

The Localized Nonlinear (LN) approximation (Habashy et al., 1993; Torres-Verdin and Habashy, 1994) is based on the assumption that the internal electrical field has a small spatial gradient, which can be neglected to zero order regardless of medium properties. As a result, the scattering tensor can be expressed in explicit form. Extending this method, we consider that variations of both electrical and magnetic fields in the vicinity of some inner point $\mathbf{r}$ of anomalous areas are sufficiently smooth:

$$\begin{array}{l}\mathbf{E}(\mathbf{r}+\delta\mathbf{r}) = \mathbf{E}(\mathbf{r}) + \delta\mathbf{r}\cdot\nabla\mathbf{E}(\mathbf{r}) + O(\delta\mathbf{r}^2),\\ \mathbf{H}(\mathbf{r}+\delta\mathbf{r}) = \mathbf{H}(\mathbf{r}) + \delta\mathbf{r}\cdot\nabla\mathbf{H}(\mathbf{r}) + O(\delta\mathbf{r}^2),\end{array} \quad \mathbf{r} \in V_\sigma, V_\mu. \tag{5.10}$$

Note that, in the last formula, we consider the dyadic product of vector operator ∇ and the vectors of electric and magnetic fields, $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$. Following Habashy et al. (1993), we obtain four scattering tensors for a model with joint electrical and magnetic inhomogeneities. Substituting expansions (5.10) into Equation (5.7) and neglecting terms of order higher than the one with respect to $\delta\mathbf{r}$, we find from Equations (5.2) and (5.7)

$$\begin{aligned}\mathbf{E} &\approx \mathbf{E}^b + \langle \hat{\mathbf{G}}^{JE}\tilde{\sigma}_a\rangle_{V_\sigma}\cdot\mathbf{E} + \langle \hat{\mathbf{G}}^{ME}\tilde{\mu}_a\rangle_{V_\mu}\cdot\mathbf{H},\\ \mathbf{H} &\approx \mathbf{H}^b + \langle \hat{\mathbf{G}}^{JH}\tilde{\sigma}_a\rangle_{V_\sigma}\cdot\mathbf{E} + \langle \hat{\mathbf{G}}^{MH}\tilde{\mu}_a\rangle_{V_\mu}\cdot\mathbf{H}.\end{aligned}$$

Introducing new compact notations and placing the background field $\mathbf{E}^{\mathrm{b}}$, $\mathbf{H}^{\mathrm{b}}$ into the right-hand side of the expressions, we obtain the following system of equations:

$$\begin{aligned}
&\mathbf{E} - \hat{\boldsymbol{\Gamma}}^{\mathrm{e}}\hat{\boldsymbol{\Pi}}^{ME} \cdot \mathbf{H} \approx \hat{\boldsymbol{\Gamma}}^{\mathrm{e}} \cdot \mathbf{E}^{\mathrm{b}}, \\
&\hat{\boldsymbol{\Gamma}}^{\mathrm{m}}\hat{\boldsymbol{\Pi}}^{JH} \cdot \mathbf{E} - \mathbf{H} \approx -\hat{\boldsymbol{\Gamma}}^{\mathrm{m}} \cdot \mathbf{H}^{\mathrm{b}},
\end{aligned} \tag{5.11}$$

where dimensionless scattering tensors are

$$\hat{\boldsymbol{\Gamma}}^{\mathrm{e}} = \left[\hat{\mathbf{I}} - \hat{\boldsymbol{\Pi}}^{JE}\right]^{-1}, \qquad \hat{\boldsymbol{\Gamma}}^{\mathrm{m}} = \left[\hat{\mathbf{I}} - \hat{\boldsymbol{\Pi}}^{MH}\right]^{-1}$$

and the integrals of the Green's dyadic functions are

$$\begin{aligned}
&\hat{\boldsymbol{\Pi}}^{JE} = \langle \hat{\mathbf{G}}^{JE}\tilde{\sigma}_{\mathrm{a}} \rangle_{V_\sigma}, \qquad \hat{\boldsymbol{\Pi}}^{ME} = \langle \hat{\mathbf{G}}^{ME}\tilde{\mu}_{\mathrm{a}} \rangle_{V_\mu}, \\
&\hat{\boldsymbol{\Pi}}^{JH} = \langle \hat{\mathbf{G}}^{JH}\tilde{\sigma}_{\mathrm{a}} \rangle_{V_\sigma}, \qquad \hat{\boldsymbol{\Pi}}^{HM} = \langle \hat{\mathbf{G}}^{MH}\tilde{\mu}_{\mathrm{a}} \rangle_{V_\mu}.
\end{aligned}$$

Applying simple linear operations to the equations in Formula (5.11), we express the anomalous fields in the form

$$\begin{aligned}
&\mathbf{E}^{\mathrm{LN}} = \hat{\boldsymbol{\Gamma}}^{JE}\mathbf{E}^{\mathrm{b}} + \hat{\boldsymbol{\Gamma}}^{ME}\mathbf{H}^{\mathrm{b}}, \\
&\mathbf{H}^{\mathrm{LN}} = \hat{\boldsymbol{\Gamma}}^{JH}\mathbf{E}^{\mathrm{b}} + \hat{\boldsymbol{\Gamma}}^{MH}\mathbf{H}^{\mathrm{b}},
\end{aligned} \qquad \mathbf{r} \in V_\sigma \cup V_\mu. \tag{5.12}$$

where $\hat{\boldsymbol{\Gamma}}^{JE}$, $\hat{\boldsymbol{\Gamma}}^{ME}$, $\hat{\boldsymbol{\Gamma}}^{JH}$ and $\hat{\boldsymbol{\Gamma}}^{MH}$ are new EM scattering tensors as introduced in (Murray et al., 1999; Cheryauka and Sato, 1999):

$$\begin{aligned}
&\hat{\boldsymbol{\Gamma}}^{JE} = \hat{\boldsymbol{\Gamma}}^{\mathrm{det}}_{\mathrm{e}}\hat{\boldsymbol{\Gamma}}^{\mathrm{e}}, \qquad \hat{\boldsymbol{\Gamma}}^{ME} = \hat{\boldsymbol{\Gamma}}^{JE}\hat{\boldsymbol{\Pi}}^{ME}\hat{\boldsymbol{\Gamma}}^{\mathrm{m}}, \\
&\hat{\boldsymbol{\Gamma}}^{MH} = \hat{\boldsymbol{\Gamma}}^{\mathrm{det}}_{\mathrm{h}}\hat{\boldsymbol{\Gamma}}^{\mathrm{m}}, \qquad \hat{\boldsymbol{\Gamma}}^{JH} = \hat{\boldsymbol{\Gamma}}^{MH}\hat{\boldsymbol{\Gamma}}^{\mathrm{e}}\hat{\boldsymbol{\Pi}}^{ME},
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\boldsymbol{\Gamma}}^{\mathrm{det}}_{\mathrm{e}} = \left[\hat{\mathbf{I}} - \hat{\boldsymbol{\Gamma}}^{\mathrm{e}}\hat{\boldsymbol{\Pi}}^{ME}\hat{\boldsymbol{\Gamma}}^{\mathrm{m}}\hat{\boldsymbol{\Pi}}^{JH}\right]^{-1}, \\
&\hat{\boldsymbol{\Gamma}}^{\mathrm{det}}_{\mathrm{h}} = \left[\hat{\mathbf{I}} - \hat{\boldsymbol{\Gamma}}^{\mathrm{m}}\hat{\boldsymbol{\Pi}}^{JH}\hat{\boldsymbol{\Gamma}}^{\mathrm{e}}\hat{\boldsymbol{\Pi}}^{ME}\right]^{-1}.
\end{aligned}$$

The fields $\mathbf{E}^{\mathrm{LN}}$, $\mathbf{H}^{\mathrm{LN}}$ outside the perturbed areas are

$$\begin{aligned}
&\mathbf{E}^{\mathrm{LN}} = \langle \hat{\mathbf{G}}^{JE}\mathbf{J}^{\mathrm{LN}} \rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{ME}\mathbf{M}^{\mathrm{LN}} \rangle_{V_\mu}, \\
&\mathbf{H}^{\mathrm{LN}} = \langle \hat{\mathbf{G}}^{JH}\mathbf{J}^{\mathrm{LN}} \rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{MH}\mathbf{M}^{\mathrm{LN}} \rangle_{V_\mu},
\end{aligned} \tag{5.13}$$

where the LN current densities $\mathbf{J}^{\mathrm{LN}}, \mathbf{M}^{\mathrm{LN}}$ are

$$\mathbf{J}^{LN} = \tilde{\sigma}_{\mathrm{a}}(\hat{\boldsymbol{\Gamma}}^{JE}\mathbf{E}^{\mathrm{b}} + \hat{\boldsymbol{\Gamma}}^{ME}\mathbf{H}^{\mathrm{b}}), \qquad \mathbf{M}^{LN} = \tilde{\mu}_{\mathrm{a}}(\hat{\boldsymbol{\Gamma}}^{JH}\mathbf{E}^{\mathrm{b}} + \hat{\boldsymbol{\Gamma}}^{MH}\mathbf{H}^{\mathrm{b}}). \tag{0.1}$$

Note that the structures of the scattering tensors $\hat{\boldsymbol{\Gamma}}^{JE}$, $\hat{\boldsymbol{\Gamma}}^{ME}$, $\hat{\boldsymbol{\Gamma}}^{JH}$, $\hat{\boldsymbol{\Gamma}}^{MH}$ can be simplified if the locations of electrical and magnetic inhomogeneities do not intersect, or the contributions of anomalous conductivity $\tilde{\sigma}_{\mathrm{a}}$ or anomalous magnetic permeability $\tilde{\mu}_{\mathrm{a}}$ dominate the other.

5. QUASI-LINEAR APPROXIMATION

Zhdanov and Fang (1996, 1997) developed a quasi-linear (QL) approximation, based on a linear relationship between anomalous and background fields inside an inhomogeneous domain, expressed by an electrical reflectivity tensor $\hat{\lambda}^{\mathrm{e}}$:

$$\mathbf{E}^{\mathrm{QL}}(\mathbf{r}) \approx \hat{\lambda}^{\mathrm{e}}(\mathbf{r})\mathbf{E}^{\mathrm{b}}(\mathbf{r}), \quad \mathbf{r} \in V_{\sigma}. \tag{5.14}$$

This tensor can be effectively approximated by a system of smooth basis functions on a coarse spatial grid because of a smooth variation of the field inside the inhomogeneity. Using a similar approach, we can introduce a 'magnetic reflectivity tensor' $\hat{\lambda}^{\mathrm{m}}$:

$$\mathbf{H}^{\mathrm{QL}}(\mathbf{r}) \approx \hat{\lambda}^{\mathrm{m}}(\mathbf{r})\mathbf{H}^{\mathrm{b}}(\mathbf{r}), \quad \mathbf{r} \in V_{\mu}. \tag{5.15}$$

According to Equations (5.7), (5.14) and (5.15), the QL approximations of the anomalous fields outside the perturbed areas are expressed by the formulae

$$\begin{aligned}\mathbf{E}^{\mathrm{QL}} &= \langle \hat{\mathbf{G}}^{JE}\mathbf{J}^{\mathrm{QL}}\rangle_{V_{\sigma}} + \langle \hat{\mathbf{G}}^{ME}\mathbf{M}^{\mathrm{QL}}\rangle_{V_{\mu}},\\ \mathbf{H}^{\mathrm{QL}} &= \langle \hat{\mathbf{G}}^{JH}\mathbf{J}^{\mathrm{QL}}\rangle_{V_{\sigma}} + \langle \hat{\mathbf{G}}^{MH}\mathbf{M}^{\mathrm{QL}}\rangle_{V_{\mu}},\end{aligned} \tag{5.16}$$

where appropriate current densities $\mathbf{J}^{\mathrm{QL}}, \mathbf{M}^{\mathrm{QL}}$ are:

$$\mathbf{J}^{\mathrm{QL}} = \tilde{\sigma}_{\mathrm{a}}(\hat{\mathbf{I}} + \hat{\lambda}^{\mathrm{e}})\mathbf{E}^{\mathrm{b}}, \qquad \mathbf{M}^{\mathrm{QL}} = \tilde{\mu}_{\mathrm{a}}(\hat{\mathbf{I}} + \hat{\lambda}^{\mathrm{m}})\mathbf{H}^{\mathrm{b}}.$$

The electrical and magnetic reflectivity tensors, $\hat{\lambda}^{\mathrm{e}}$ and $\hat{\lambda}^{\mathrm{m}}$, are determined by a minimization technique applied to the corresponding areas of anomalous conductivity support, V_{σ}, and anomalous magnetic permeability support, V_{μ}, according to the following equations.

(1) Within the joint area of the domains V_{σ} and V_{μ}, $\mathbf{r} \in V = V_{\sigma} \bigcap V_{\mu}$

$$\left\| \begin{array}{l} \hat{\lambda}^{\mathrm{e}}\mathbf{E}^{\mathrm{b}} - \langle \hat{\mathbf{G}}^{JE}\tilde{\sigma}_{\mathrm{a}}(\mathbf{I} + \hat{\lambda}^{\mathrm{e}})\mathbf{E}^{\mathrm{b}} + \hat{\mathbf{G}}^{ME}\tilde{\mu}_{\mathrm{a}}(\hat{\mathbf{I}} + \hat{\lambda}^{\mathrm{m}})\mathbf{H}^{\mathrm{b}}\rangle_{V} \\ \hat{\lambda}^{\mathrm{m}}\mathbf{H}^{\mathrm{b}} - \langle \hat{\mathbf{G}}^{JH}\tilde{\sigma}_{\mathrm{a}}(\mathbf{I} + \hat{\lambda}^{\mathrm{e}})\mathbf{E}^{\mathrm{b}} + \hat{\mathbf{G}}^{MH}\tilde{\mu}_{\mathrm{a}}(\hat{\mathbf{I}} + \hat{\lambda}^{\mathrm{m}})\mathbf{H}^{\mathrm{b}}\rangle_{V} \end{array} \right\| = \min. \tag{5.17}$$

(2) Within the area outside V_{μ} but inside V_{σ}, $\mathbf{r} \in V = V_{\sigma} \setminus V_{\mu}$

$$\left\| \hat{\lambda}^{\mathrm{e}}\mathbf{E}^{\mathrm{b}} - \langle \hat{\mathbf{G}}^{JE}\tilde{\sigma}_{\mathrm{a}}(\hat{\mathbf{I}} + \hat{\lambda}^{\mathrm{e}})\mathbf{E}^{\mathrm{b}}\rangle_{V} \right\| = \min. \tag{5.18}$$

(3) Within the area outside V_{σ} but inside V_{μ}, $\mathbf{r} \in V = V_{\mu} \setminus V_{\sigma}$

$$\left\| \hat{\lambda}^{\mathrm{m}}\mathbf{H}^{\mathrm{b}} - \langle \hat{\mathbf{G}}^{MH}\tilde{\mu}_{\mathrm{a}}(\hat{\mathbf{I}} + \hat{\lambda}^{\mathrm{m}})\mathbf{H}^{\mathrm{b}}\rangle_{V} \right\| = \min. \tag{5.19}$$

Note that the solution of Equations (5.17)–(5.19) is nonlinear with respect to $\tilde{\sigma}_{\mathrm{a}}$ and $\tilde{\mu}_{\mathrm{a}}$, because $\hat{\lambda}^{\mathrm{e}}$ and $\hat{\lambda}^{\mathrm{m}}$ are nonlinear functions of $\tilde{\sigma}_{\mathrm{a}}$ and $\tilde{\mu}_{\mathrm{a}}$.

6. QUASI-ANALYTICAL APPROXIMATION

The Quasi-Analytical (QA) approximation (Zhdanov et al., 2000) is based on the same assumption as the QL approximation that the anomalous fields inside an inhomogeneous domain are linearly proportional to the background EM fields through the reflectivity tensors $\hat{\lambda}^e$ and $\hat{\lambda}^m$ (Equations (5.14) and (5.15)). The main difference is that, using an analytic technique in the QA approximation, Zhdanov et al. (2000) obtained the reflectivity tensor $\hat{\lambda}^e$ in explicit form. Here we extend the QA approach for media with joint electrical and magnetic inhomogeneities.

EM fields at inner points can be expressed using Equations (5.14) and (5.15) as

$$\hat{\lambda}^e\mathbf{E}^b = \mathbf{E}^B + \langle \hat{\mathbf{G}}^{JE}\tilde{\sigma}_a\hat{\lambda}^e\mathbf{E}^b\rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{ME}\tilde{\mu}_a\hat{\lambda}^m\mathbf{H}^b\rangle_{V_\mu},$$
$$\hat{\lambda}^m\mathbf{H}^b = \mathbf{H}^B + \langle \hat{\mathbf{G}}^{JH}\tilde{\sigma}_a\hat{\lambda}^e\mathbf{E}^b\rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{MH}\tilde{\mu}_a\hat{\lambda}^m\mathbf{H}^b\rangle_{V_\mu}, \tag{5.20}$$

where $\mathbf{E}^B, \mathbf{H}^B$ are anomalous Born fields:

$$\mathbf{E}^B = \langle \hat{\mathbf{G}}^{JE}\tilde{\sigma}_a\mathbf{E}^b\rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{ME}\tilde{\mu}_a\mathbf{H}^b\rangle_{V_\mu} = \mathbf{E}^{B\sigma} + \mathbf{E}^{B\mu},$$
$$\mathbf{H}^B = \langle \hat{\mathbf{G}}^{JH}\tilde{\sigma}_a\mathbf{E}^b\rangle_{V_\sigma} + \langle \hat{\mathbf{G}}^{MH}\tilde{\mu}_a\mathbf{H}^b\rangle_{V_\mu} = \mathbf{H}^{B\sigma} + \mathbf{H}^{B\mu}. \tag{5.21}$$

Subtracting weighted Born fields from Equation (5.20), we obtain

$$\hat{\lambda}^e(\mathbf{E}^b - \mathbf{E}^{B\sigma}) - \hat{\lambda}^m\mathbf{E}^{B\mu} = \mathbf{E}^B + \langle (\hat{\mathbf{G}}^{JE}\tilde{\sigma}_a\hat{\lambda}^e(\mathbf{r}') - \hat{\lambda}^e(\mathbf{r})\hat{\mathbf{G}}^{JE}\tilde{\sigma}_a)\mathbf{E}^b\rangle_{V_\sigma}$$
$$+ \langle (\hat{\mathbf{G}}^{ME}\tilde{\mu}_a\hat{\lambda}^m(\mathbf{r}') - \hat{\lambda}^m(\mathbf{r})\hat{\mathbf{G}}^{ME}\tilde{\mu}_a)\mathbf{H}^b\rangle_{V_\mu},$$
$$-\hat{\lambda}^e\mathbf{H}^{B\sigma} + \hat{\lambda}^m(\mathbf{H}^b - \mathbf{H}^{B\mu}) = \mathbf{H}^B + \langle (\hat{\mathbf{G}}^{JH}\tilde{\sigma}_a\hat{\lambda}^e(\mathbf{r}') - \hat{\lambda}^e(\mathbf{r})\hat{\mathbf{G}}^{JH}\tilde{\sigma}_a)\mathbf{E}^b\rangle_{V_\sigma}$$
$$+ \langle (\hat{\mathbf{G}}^{MH}\tilde{\mu}_a\hat{\lambda}^m(\mathbf{r}') - \hat{\lambda}^m(\mathbf{r})\hat{\mathbf{G}}^{MH}\tilde{\mu}_a)\mathbf{H}^b\rangle_{V_\mu}. \tag{5.22}$$

Following Habashy et al. (1993) and Torres-Verdin and Habashy (1994), we can take into account that the Green's tensors $\hat{\mathbf{G}}^{JE}$, $\hat{\mathbf{G}}^{ME}$, $\hat{\mathbf{G}}^{JH}$ and $\hat{\mathbf{G}}^{MH}$ exhibit either a singularity or a peak at the point $\mathbf{r}_j = \mathbf{r}$. Therefore, one can expect that the dominant contribution to the integral in Equation (5.22) is from some vicinity of the point $\mathbf{r}_j = \mathbf{r}$. Assuming also that $\hat{\lambda}^e$ and $\hat{\lambda}^m$ are the slowly varying functions within domains V_σ, V_μ one can rewrite Equation (5.22) in the form

$$\hat{\lambda}^e(\mathbf{E}^b - \mathbf{E}^{B\sigma}) - \hat{\lambda}^m\mathbf{E}^{B\mu} \approx \mathbf{E}^B, \tag{5.23}$$

$$-\hat{\lambda}^e\mathbf{H}^{B\sigma} + \hat{\lambda}^m(\mathbf{H}^b - \mathbf{H}^{B\mu}) \approx \mathbf{H}^B. \tag{5.24}$$

Note that the system of Equations (5.23) and (5.24) is, in general cases, underdetermined, because we have two vector equations for two unknown tensors $\hat{\lambda}^e$ and $\hat{\lambda}^m$. Let us consider now two special cases with scalar and diagonal reflectivity tensors.

In the case of scalar reflectivity tensors ($\hat{\lambda}^e = \lambda^e\hat{\mathbf{I}}$, $\hat{\lambda}^m = \lambda^m\hat{\mathbf{I}}$, where $\hat{\mathbf{I}}$ is a unit tensor), the linear system of Equations (5.23) and (5.24) is overdetermined. We can obtain two scalar equations for λ^e and λ^m by choosing the specific type of the multipliers. In particular, we calculate first, the dot product of both sides of Equation (5.23) and the complex conjugate electric field $\mathbf{E}^{b*}$, and, second, calculate the dot product of both sides of Equation (5.24), the complex conjugate magnetic field $\mathbf{H}^{b*}$:

$$\lambda^e(\mathbf{E}^b \cdot \mathbf{E}^{b*} - \mathbf{E}^{B\sigma} \cdot \mathbf{E}^{b*}) - \lambda^m\mathbf{E}^{B\mu} \cdot \mathbf{E}^{b*} \approx \mathbf{E}^B \cdot \mathbf{E}^{b*}, \tag{5.25}$$

$$-\lambda^e\mathbf{H}^{B\sigma} \cdot \mathbf{H}^{b*} + \lambda^m(\mathbf{H}^b \cdot \mathbf{H}^{b*} - \mathbf{H}^{B\mu} \cdot \mathbf{H}^{b*}) \approx \mathbf{H}^B \cdot \mathbf{H}^{b*}, \tag{5.26}$$

where '*' denotes the complex conjugate vectors.

As a result, we obtain a system of two linear equations with respect to $\hat{\lambda}^{\mathrm{e}}$ and $\hat{\lambda}^{\mathrm{m}}$:

$$\begin{pmatrix} \mathbf{E}^{\mathrm{b}}\cdot\mathbf{E}^{\mathrm{b}*}-\mathbf{E}^{\mathrm{B}\sigma}\cdot\mathbf{E}^{\mathrm{b}*} & -\mathbf{E}^{\mathrm{B}\mu}\cdot\mathbf{E}^{\mathrm{b}*} \\ -\mathbf{H}^{\mathrm{B}\sigma}\cdot\mathbf{H}^{\mathrm{b}*} & \mathbf{H}^{\mathrm{b}}\cdot\mathbf{H}^{\mathrm{b}*}-\mathbf{H}^{\mathrm{B}\mu}\cdot\mathbf{H}^{\mathrm{b}*} \end{pmatrix}\begin{pmatrix} \lambda^{\mathrm{e}} \\ \lambda^{\mathrm{m}} \end{pmatrix}=\begin{pmatrix} \mathbf{E}^{\mathrm{B}}\cdot\mathbf{E}^{\mathrm{b}*} \\ \mathbf{H}^{\mathrm{B}}\cdot\mathbf{H}^{\mathrm{b}*} \end{pmatrix}, \tag{5.27}$$

which can be easily resolved as follows:

$$\lambda^{\mathrm{e}}=D^{-1}\left[\left(\mathbf{E}^{\mathrm{B}}\cdot\mathbf{E}^{\mathrm{b}*}\right)\left(\mathbf{H}^{\mathrm{b}}\cdot\mathbf{H}^{\mathrm{b}*}-\mathbf{H}^{\mathrm{B}\mu}\cdot\mathbf{H}^{\mathrm{b}*}\right)+\left(\mathbf{E}^{\mathrm{B}\mu}\cdot\mathbf{E}^{\mathrm{b}*}\right)\left(\mathbf{H}^{\mathrm{B}}\cdot\mathbf{H}^{\mathrm{b}*}\right)\right],$$

$$\lambda^{\mathrm{m}}=D^{-1}\left[\left(\mathbf{H}^{\mathrm{B}}\cdot\mathbf{H}^{\mathrm{b}*}\right)\left(\mathbf{E}^{\mathrm{b}}\cdot\mathbf{E}^{\mathrm{b}*}-\mathbf{E}^{\mathrm{B}\sigma}\cdot\mathbf{E}^{\mathrm{b}*}\right)+\left(\mathbf{H}^{\mathrm{B}\sigma}\cdot\mathbf{H}^{\mathrm{b}*}\right)\left(\mathbf{E}^{\mathrm{B}}\cdot\mathbf{E}^{\mathrm{b}*}\right)\right], \tag{5.28}$$

assuming that the determinant, D, of the matrix of the linear Equation (5.27) is not equal to zero:

$$D=\left(\mathbf{E}^{\mathrm{b}}\cdot\mathbf{E}^{\mathrm{b}*}-\mathbf{E}^{\mathrm{B}\sigma}\cdot\mathbf{E}^{\mathrm{b}*}\right)\left(\mathbf{H}^{\mathrm{b}}\cdot\mathbf{H}^{\mathrm{b}*}-\mathbf{H}^{\mathrm{B}\mu}\cdot\mathbf{H}^{\mathrm{b}*}\right)-\left(\mathbf{E}^{\mathrm{B}\mu}\cdot\mathbf{E}^{\mathrm{b}*}\right)\left(\mathbf{H}^{\mathrm{B}\sigma}\cdot\mathbf{H}^{\mathrm{b}*}\right)\neq 0.$$

For example, in the case of the purely electrical inhomogeneities,

$$\mathbf{E}^{\mathrm{B}\mu}=0, \qquad \mathbf{H}^{\mathrm{B}\mu}=0,$$
$$\mathbf{E}^{\mathrm{B}}=\mathbf{E}^{\mathrm{B}\sigma}, \qquad \mathbf{H}^{\mathrm{B}}=\mathbf{H}^{\mathrm{B}\sigma},$$

and the formula for the electrical reflectivity coefficient becomes

$$\lambda^{\mathrm{e}}=\frac{\left(\mathbf{E}^{\mathrm{B}\sigma}\cdot\mathbf{E}^{\mathrm{b}*}\right)}{\left(\mathbf{E}^{\mathrm{b}}\cdot\mathbf{E}^{\mathrm{b}*}-\mathbf{E}^{\mathrm{B}\sigma}\cdot\mathbf{E}^{\mathrm{b}*}\right)}. \tag{5.29}$$

Formula (5.29) is equivalent to the one developed in Zhdanov et al. (2000). Note that by choosing different multipliers in Equations (5.25) and (5.26) one can select various situations for reflectivity coefficients.

In the case of diagonal reflectivity tensors $\hat{\lambda}^{\mathrm{e}}$, $\hat{\lambda}^{\mathrm{m}}$, we obtain from Equations (5.23) and (5.24) a 6×6 system of linear equations with respect to the six unknown diagonal components of tensors $\lambda^{\mathrm{e}}_{ii}$, $\lambda^{\mathrm{m}}_{ii}$, $i=1,2,3$:

$$\begin{pmatrix} E_1^b-E_1^{B\sigma} & 0 & 0 & -E_1^{B\mu} & 0 & 0 \\ 0 & E_2^b-E_2^{B\sigma} & 0 & 0 & -E_2^{B\mu} & 0 \\ 0 & 0 & E_3^b-E_3^{B\sigma} & 0 & 0 & -E_3^{B\mu} \\ -H_1^{B\sigma} & 0 & 0 & H_1^b-H_1^{B\mu} & 0 & 0 \\ 0 & -H_2^{B\sigma} & 0 & 0 & H_2^b-H_2^{B\mu} & 0 \\ 0 & 0 & -H_3^{B\sigma} & 0 & 0 & H_3^b-H_3^{B\mu} \end{pmatrix}$$

$$\times\begin{pmatrix} \lambda^{\mathrm{e}}_{11} \\ \lambda^{\mathrm{e}}_{22} \\ \lambda^{\mathrm{e}}_{33} \\ \lambda^{\mathrm{m}}_{11} \\ \lambda^{\mathrm{m}}_{22} \\ \lambda^{\mathrm{m}}_{33} \end{pmatrix}=\begin{pmatrix} E_1^{\mathrm{B}} \\ E_2^{\mathrm{B}} \\ E_3^{\mathrm{B}} \\ H_1^{\mathrm{B}} \\ H_2^{\mathrm{B}} \\ H_3^{\mathrm{B}} \end{pmatrix}. \tag{5.30}$$

To solve this sparse problem we consider separately three pairs of equations: the 1st and the 4th, the 2nd and the 5th, and the 3rd and the 6th:

$$\begin{cases} (E_i^{\mathrm{b}}-E_i^{\mathrm{B}\sigma})\lambda^{\mathrm{e}}_{ii}-E_i^{\mathrm{B}\mu}\lambda^{\mathrm{m}}_{ii}=E_i^{\mathrm{B}} \\ -H_i^{\mathrm{B}\sigma}\lambda^{\mathrm{e}}_{ii}+(H_i^{\mathrm{b}}-H_i^{\mathrm{B}\mu})\lambda^{\mathrm{m}}_{ii}=H_i^{\mathrm{B}} \end{cases} \quad i=1,2,3. \tag{5.31}$$

Solving each of the 2×2 equations from the system (5.31) under the assumption that the determinant, D_i, of the corresponding matrix of the linear equations (5.31) is not equal to zero,

$$D_i = (E_i^{\mathrm{b}} - E_i^{\mathrm{B}\sigma})(H_i^{\mathrm{b}} - H_i^{\mathrm{B}\mu}) - H_i^{\mathrm{B}\sigma} E_i^{\mathrm{B}\mu} \neq 0,$$

we obtain the values of the diagonal tensor components $\lambda_{ii}^{\mathrm{e}}$ and $\lambda_{ii}^{\mathrm{m}}$

$$\begin{aligned} \lambda_{ii}^{\mathrm{e}} &= D_i^{-1}\left[E_i^{\mathrm{B}}(H_i^{\mathrm{b}} - H_i^{\mathrm{B}\mu}) + H_i^{\mathrm{B}} E_i^{\mathrm{B}\mu}\right], \\ & \qquad\qquad\qquad\qquad i = 1,2,3. \\ \lambda_{ii}^{\mathrm{m}} &= D_i^{-1}\left[H_i^{\mathrm{B}}(E_i^{\mathrm{b}} - E_i^{\mathrm{B}\sigma}) + E_i^{\mathrm{B}} H_i^{\mathrm{B}\sigma}\right]. \end{aligned} \tag{5.32}$$

Finally, for purely electrical inhomogeneities, we find

$$\lambda_{ii}^{\mathrm{e}} = \frac{E_i^{\mathrm{B}\sigma}}{E_i^{\mathrm{b}} - E_i^{\mathrm{B}\sigma}}, \quad i = 1,2,3, \tag{5.33}$$

while for purely magnetic inhomogeneities we have

$$\lambda_{ii}^{\mathrm{m}} = \frac{H_i^{\mathrm{B}\mu}}{H_i^{\mathrm{b}} - H_i^{\mathrm{B}\mu}}, \quad i = 1,2,3. \tag{5.34}$$

7. NUMERICAL RESULTS AND VALIDATION

Let us consider now several numerical applications of the developed approximations.

In the first set of numerical experiments, we check the validity of the linear and nonlinear approximations in a one-dimensional cylindrically layered model, excited by a vertical magnetic dipole (Figure 2). The EM induction response in the cylindrically layered model can be calculated using an integral equation method. In Appendix A we

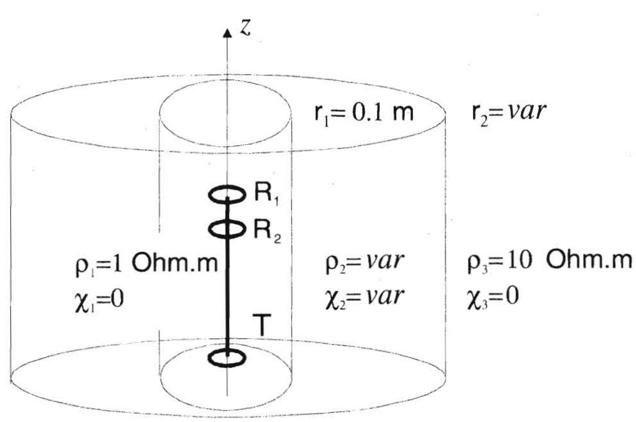

Figure 2. One-dimensional cylindrically layered model and induction array. The characteristics of the three-coil induction tool $TxRx_1Rx_2$ are as follows: $f = 50$ kHz; $L_1 = 1.5$ m, $L_2 = 1$ m; $M_1 = 1$ A m^2; $M_2 = -0.2963$ A m^2, where f is the frequency, L_1 and L_2 are the tool spacings, and M_1 and M_2 are the receiver moments.

implement a quasi-analytical approach to computing the EM field in the model with axial symmetry and present the corresponding formulae. In our simple computational test we consider the second layer in the three-layered model as an anomalous domain (scatterer) with respect to the two-layer model 'borehole space–background formation', chosen as a background model. This model study simulates an invaded zone effect in a thick formation, which is a typical problem in induction logging.

At the same time, electromagnetic fields in cylindrically layered models with a radial piecewise distribution of electromagnetic parameters can be expressed in closed integral form and can be calculated with arbitrary accuracy. The mathematical description and the solution for the vertical component of the magnetic field can be found, for instance, in Augustin et al. (1989). We treat the result obtained by using this calculation technique as an 'exact' solution. We analyze a synthetic voltage signal of a borehole compensated array with characteristics close to practical ones. The voltage signal of the conventional three-coil differential induction tool is represented as a linear combination of the two-coil tool's responses (Kaufman and Keller, 1989):

$$V = V_1 + V_2 = \tilde{\mu}_1 \left[M_1 H^0_{zz}(L_1) + M_2 H^0_{zz}(L_2) \right] + \frac{\tilde{\mu}_1}{2\pi^2} \int_0^\infty p_1^2 \nu_1(\lambda)(M_1 \cos \lambda L_1 + M_2 \cos \lambda L_2) \mathrm{d}\lambda,$$

where $\{V_1, M_1, L_1\}$, $\{V_2, M_2, L_2\}$ are the voltage signals, the coil moments, and the spacings of the 1st and the 2nd two-coil induction subarrays; $\nu_1(\lambda)$ is the so-called 'layered function' within the borehole space and H^0_{zz} is a primary field in the homogeneous medium with the parameters of the borehole space $k_1^2 = \tilde{\mu}_1 \tilde{\sigma}_1$, $p_1^2 = \lambda^2 - k_1^2$.

Figure 3 shows the real and imaginary parts of the voltage (R and X signals in logging terminology), calculated by using the exact data (the closed form solution: real part, solid line; imaginary part, dotted line) and the combined approximation (real part, '+' symbols; imaginary part, 'o' symbols). As one can find from Equations (5.6) and (5.7), the anomalous field components are determined by the superposition of electric and magnetic scattering currents depending on fluctuations of the material properties. We can separately choose a form of the approximation for these currents and compose a hybrid type approximation. Here we implement the quasi-analytical approximation for the anomalous electrical resistivity ϱ and the Born approximation for the anomalous magnetic susceptibility χ.

We compute three-coil induction tool responses for three cases:

(1) a variable resistivity ρ_2 of the second cylindrical layer with a fixed magnetic susceptibility $\chi_2 = 0.01$ and the radius of the outer boundary $r_2 = 1.5$ m (Figure 3a);

(2) a variable susceptibility χ_2 with a fixed resistivity $\rho_2 = 2$ Ohm m and the radius of the outer boundary $r_2 = 1.5$ m (Figure 3b); and

(3) a variable radius of the outer boundary r_2 of the second cylindrical layer with a fixed resistivity $\rho_2 = 2$ Ohm m and magnetic susceptibility $\chi_2 = 0.01$ (Figure 3c).

Our calculations demonstrate that the Born approximation provides a reasonable response estimation in a model with variable magnetic susceptibility, because the range of susceptibility variations in the invaded formations is small. At the same time, the electrical resistivity can change by a factor up four orders of magnitude, and one has

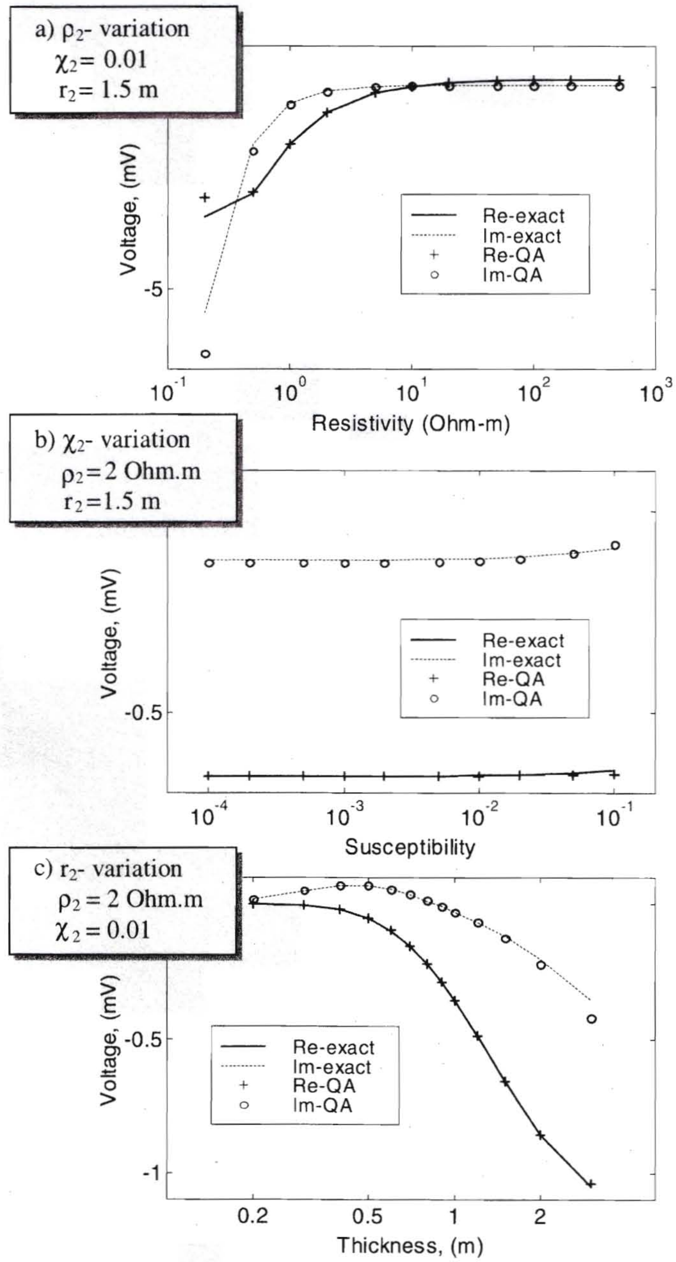

Figure 3. The comparison of the synthetic voltage signals. The responses were calculated using the closed form solution (real part, solid line; imaginary part, dashed line) and the combined QA approximation for the electrical resistivity ϱ_2 and Born approximation for the magnetic susceptibility χ_2 (real part, '+' symbols; imaginary part, 'o' symbols). (a) Variation of the layer electrical resistivity. (b) Variation of the layer magnetic susceptibility. (c) Variation of the layer outer boundary.

to apply the nonlinear approximations (in our case we use the QA approximation) for high-contrast and large-size anomalous areas.

In the second set of numerical simulations, we study the ability of the nonlinear methods to simulate an EM field in borehole models with casing. These models have extremely high contrasts in electrical conductivity and magnetic susceptibility parameters. For instance, the mild steel, which is widely used for borehole casing, has electrical resistivity in the order of 10^{-6} Ohm m and magnetic susceptibility in the order of 10^3 (Balasnis, 1989).

Figure 4 demonstrates the results of the comparison between exact, localized and quasi-analytical solutions. The approximate data produced by both nonlinear methods for anomalous fields have good accuracy and graphically fit the exact data. At the same time, the quasi-analytical formulation gives the better approximation, because it takes into account the contribution from scattering currents depending on a primary source location.

For this model we study also the validity of casing approximation using well known electrical (S) and magnetic (M) thin sheets models (Figures 5 and 6). In the general case of electrical and magnetic inhomogeneities and arbitrary polarization of a primary source the EM fields are functions not only of wave numbers k_i of a medium, but they also depend on ratios (contrasts) of electrical and magnetic properties (Felsen and Marcuvitz, 1994). Thus, we plan to consider the quality of the approximate solutions and the effects of anomalous electrical conductivity and magnetic susceptibility separately.

We have found that the casing can be considered as an S thin sheet, which is characterized by a specific conductance (Figure 5):

$$S = \frac{1}{\rho_2}\,\mathrm{d}r = \mathrm{const.}$$

and low-magnetic properties only. The real thickness of the casing, $\mathrm{d}r$, may vary from 0.001 m to 0.05 m (with the corresponding change of the conductivity $1/\rho_2$ to keep the conductance constant) without significant effect in the induction tool response for materials of low-magnetic susceptibility value, $\chi_2 \leq 1$, (upper panel, Figure 5). Highly magnetized casing with $\chi_2 = 10$ to 10^3 (Figure 5, middle and bottom panels) cannot be satisfactory simulated by S thin-sheet approximation.

A similar effect is observed for an M thin sheet with a resistivity $\rho_2 = 10^{-1}$ Ohm m and with a constant integrated magnetic susceptibility (Figure 6, upper panel):

$$M = (1 + \chi_2)\,\mathrm{d}r = \mathrm{const.}$$

However, this equivalence is not perfect for a lower resistivity of 10^{-3} Ohm m (Figure 6, middle panel); and one cannot neglect the casing thickness for a highly conductive casing with $\rho_2 = 10^{-6}$ Ohm m (Figure 6, bottom panel).

8. CONCLUSIONS

In this paper we have introduced a family of nonlinear approximations of electromagnetic field in models with joint electrical and magnetic inhomogeneities. These approximations are presented in the form of tensor integrals over a domain with anomalous

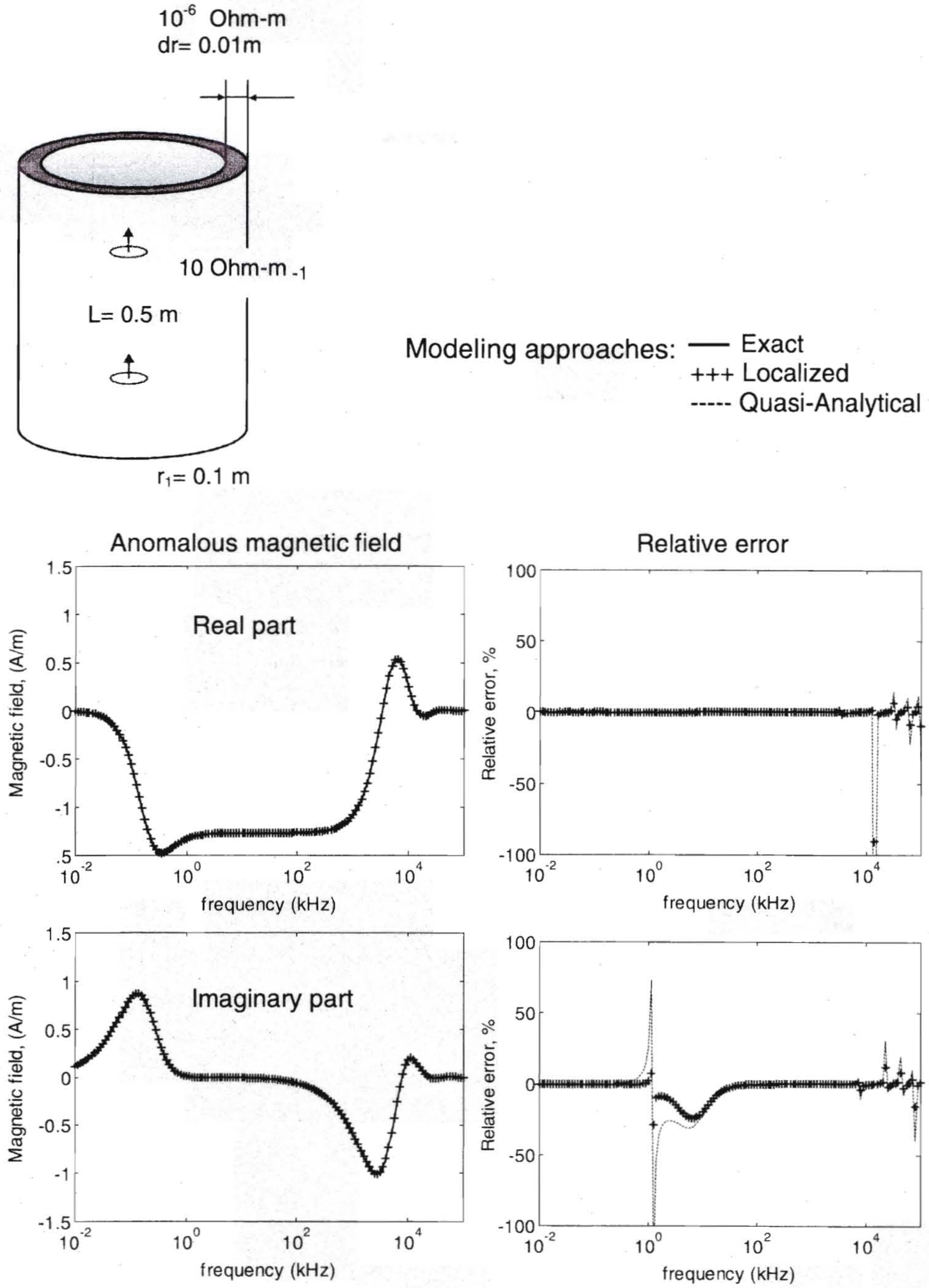

Figure 4. The comparison of the exact, LN, and QA solutions in the casing model.

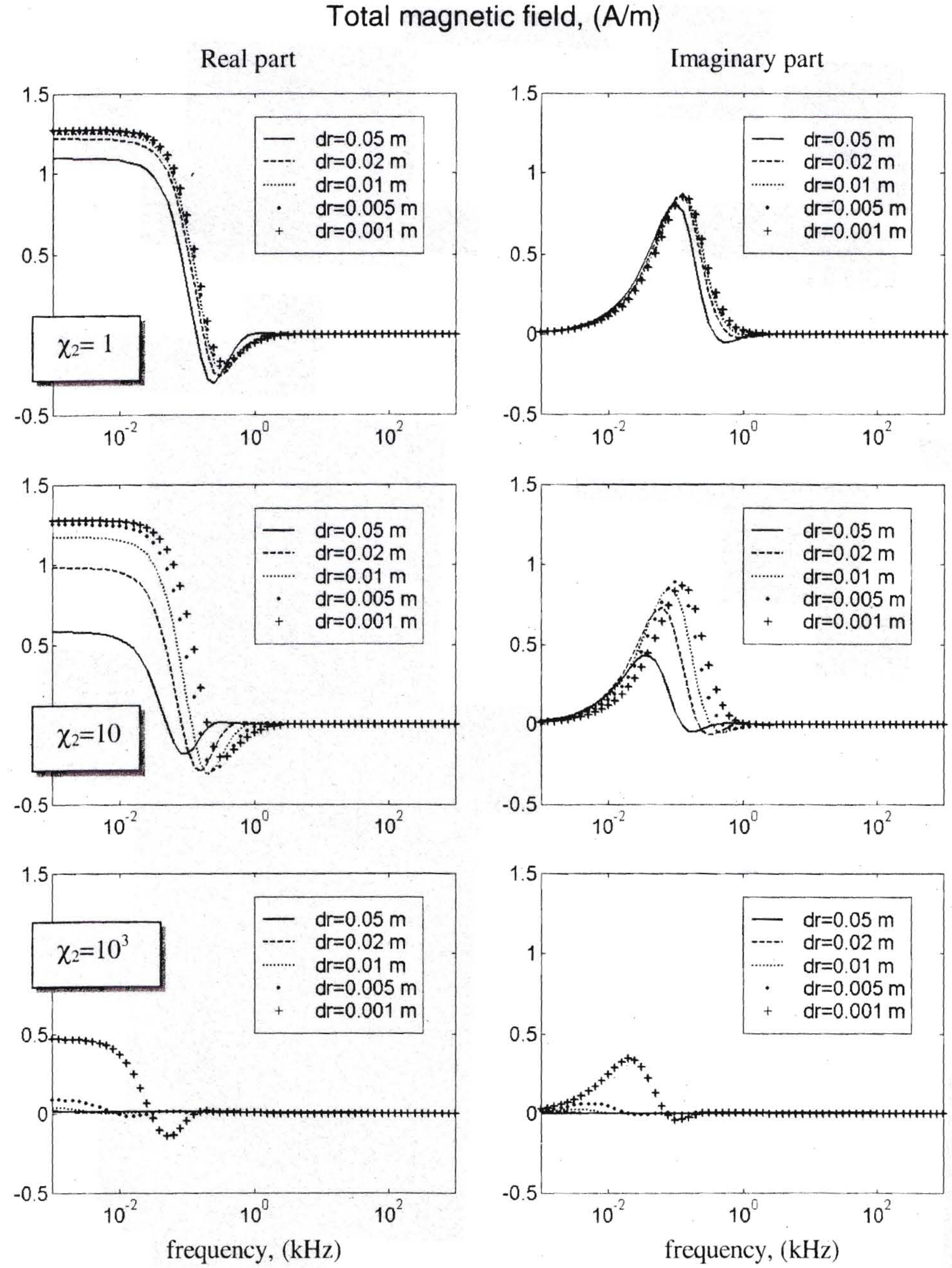

Figure 5. *S*-equivalence in the casing model, $\rho_2^{-1}\,dr = 10^4\,S$.

parameters. The developed approximations combine linear and nonlinear estimations depending on the range of the complex conductivity and magnetic susceptibility perturbations. The introduced family of nonlinear approximations could form a basis for

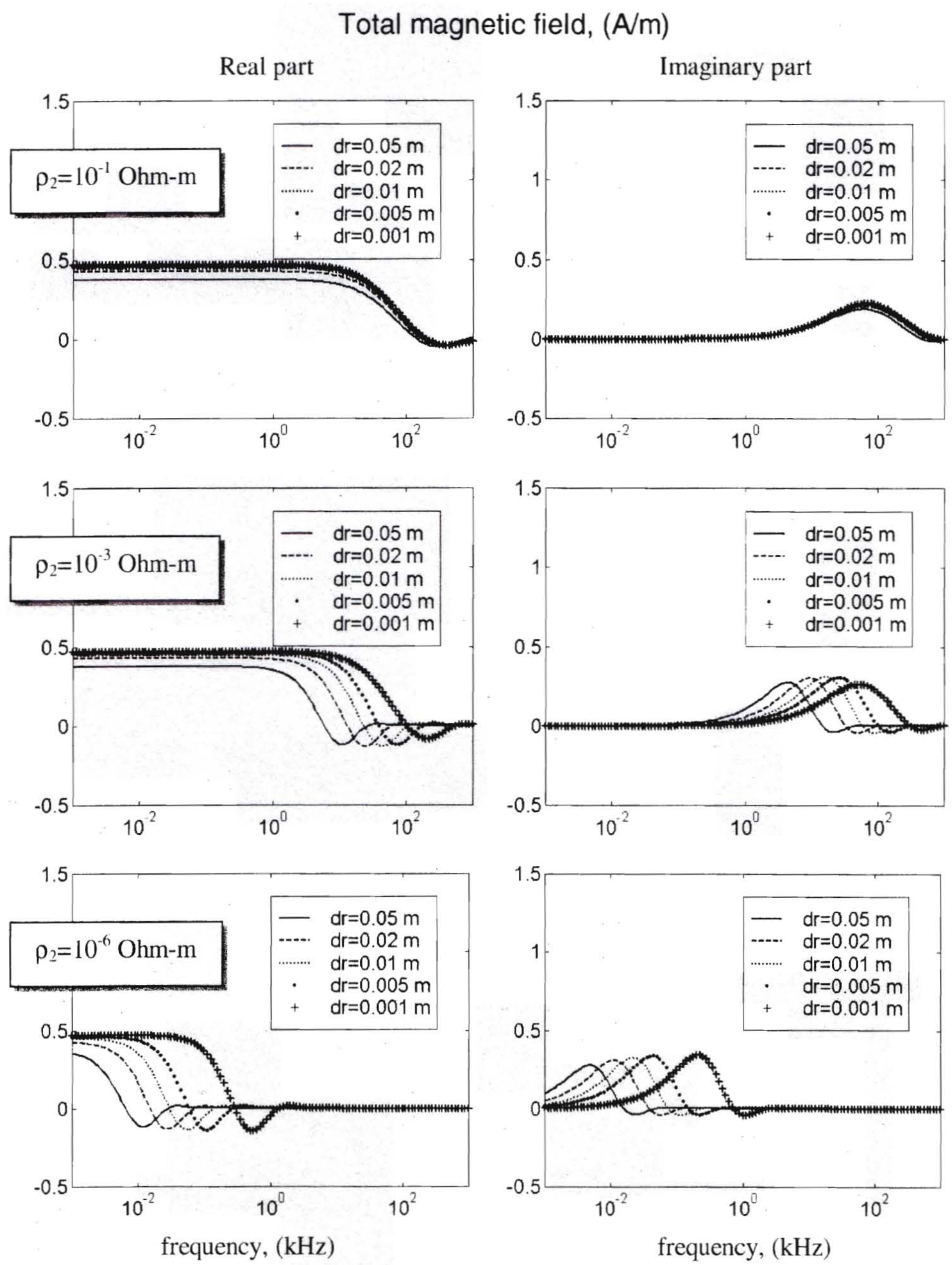

Figure 6. ***M*-equivalence in the casing model, $(1+\chi_2)\,dr = 1M$.**

fast EM modeling and imaging in the multi-dimensional environment where the joint electrical and magnetic inhomogeneities are the essential feature of the model. The numerical tests carried out for one-dimensional electromagnetic logging application have demonstrated the validity of the theory and the effectiveness of the proposed approximations.

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Appendix A. QUASI-ANALYTICAL ONE-DIMENSIONAL SOLUTION

Consider a one-dimensional cylindrically layered model with the cylindrical coordinate system $\{\rho, \phi, z\}$ and the axial distribution of electrical conductivity $\tilde{\sigma}(\rho)$ and magnetic permeability $\tilde{\mu}(\rho)$. The incident field is excited by the vertical magnetic dipole of a unit moment located at the axis of symmetry at the point z_p . We formulate a simple axial symmetrical one-dimensional boundary value problem, where the model properties do not depend on the ϕ and z coordinates and the EM field is also azimuthally uniform.

Applying this Fourier transform with respect to the variable z to the initial Equations (5.6) and (5.7) replacing all the functions $\mathbf{E}, \mathbf{H}, \hat{\mathbf{G}}$ with their appropriate spectra $\mathbf{e}, \mathbf{h}, \hat{\mathbf{g}}$, we obtain

$$\mathbf{e}^{\mathrm{a}} = \langle \hat{\mathbf{g}}^{JE} \tilde{\sigma}_{\mathrm{a}} (\mathbf{e}^{\mathrm{b}} + \mathbf{e}^{\mathrm{a}}) \rangle_{V_\sigma(\rho', \phi')} + \langle \hat{\mathbf{g}}^{ME} \tilde{\mu}_{\mathrm{a}} (\mathbf{h}^{\mathrm{b}} + \mathbf{h}^{\mathrm{a}}) \rangle_{V_\mu(\rho', \phi')},$$
$$\mathbf{h}^{\mathrm{a}} = \langle \hat{\mathbf{g}}^{JH} \tilde{\sigma}_{\mathrm{a}} (\mathbf{e}^{\mathrm{b}} + \mathbf{e}^{\mathrm{a}}) \rangle_{V_\sigma(\rho', \phi')} + \langle \hat{\mathbf{g}}^{MH} \tilde{\mu}_{\mathrm{a}} (\mathbf{h}^{\mathrm{b}} + \mathbf{h}^{\mathrm{a}}) \rangle_{V_\mu(\rho', \phi')}. \tag{5.A1}$$

Defining Green's tensor functions, $\check{\mathbf{G}}$, of the circular source by integration of the Green's tensor functions $\hat{\mathbf{G}}$ of the point sources as

$$\check{\mathbf{G}} = \int_0^{2\pi} \hat{\mathbf{G}}(\phi') d\phi', \tag{5.A2}$$

and substituting their spectra $\check{\mathbf{g}}^{\alpha\beta}$ in Equation (5.A1), we obtain

$$\mathbf{e}^{\mathrm{a}} = \langle \check{\mathbf{g}}^{DE} \tilde{\sigma}_{\mathrm{a}} (\mathbf{e}^{\mathrm{b}} + \mathbf{e}^{\mathrm{a}}) \rangle_{V_\sigma(\rho')} + \langle \check{\mathbf{g}}^{ME} \tilde{\mu}_{\mathrm{a}} (\mathbf{h}^{\mathrm{b}} + \mathbf{h}^{\mathrm{a}}) \rangle_{V_\mu(\rho')},$$
$$\mathbf{h}^{\mathrm{a}} = \langle \check{\mathbf{g}}^{JH} \tilde{\sigma}_{\mathrm{a}} (\mathbf{e}^{\mathrm{b}} + \mathbf{e}^{\mathrm{a}}) \rangle_{V_\sigma(\rho')} + \langle \check{\mathbf{g}}^{MH} \tilde{\mu}_{\mathrm{a}} (\mathbf{h}^{\mathrm{b}} + \mathbf{h}^{\mathrm{a}}) \rangle_{V_\mu(\rho')}. \tag{5.A3}$$

According to Equation (5.5), the tensors $\check{\mathbf{g}}^{JE}$, $\check{\mathbf{g}}^{JH}$, $\check{\mathbf{g}}^{MH}$, $\check{\mathbf{g}}^{ME}$ satisfy the following equations:

$$\begin{aligned}
&(\nabla^* \times \frac{1}{\tilde{\mu}_b(\rho)}\nabla^* \times -\tilde{\sigma}_b(\rho))\check{\mathbf{g}}^{JE}(\rho'|\rho) = \hat{\mathbf{I}}\frac{\delta(\rho-\rho')}{\rho},\\
&\check{\mathbf{g}}^{JH}(\rho'|\rho) = \frac{1}{\tilde{\mu}_b(\rho')}\nabla^* \times \check{\mathbf{g}}^{JE}(\rho'|\rho),\\
&(\nabla^* \times \frac{1}{\tilde{\sigma}_b(\rho)}\nabla^* \times -\tilde{\mu}_b(\rho))\check{\mathbf{g}}^{MH}(\rho'|\rho) = \hat{\mathbf{I}}\frac{\delta(\rho-\rho')}{\rho},\\
&\check{\mathbf{g}}^{ME}(\rho'|\rho) = \frac{0}{\tilde{\sigma}_b(\rho')}\nabla^* \times \check{\mathbf{g}}^{MH}(\rho'|\rho),
\end{aligned} \tag{5.A4}$$

where $\nabla^* = \frac{\partial}{\partial\rho}\mathbf{i}_\rho + i\lambda\mathbf{i}_z$ is a Fourier spectrum of the axial symmetrical vector-operator ∇.

Equations (5.A4) for the tensor functions $\check{\mathbf{g}}^{\alpha\beta}$ in the cylindrically layered model can be solved in close analytical form. However, we do not present here the general analytical solutions because of the tremendous length and complexity of these expressions. We restrict our study to the simplest case of the homogeneous model only. In this case, the solutions of Equations (5.A4) can be expressed in the compact form

$$\begin{aligned}
&\check{\mathbf{g}}^{JE}(\rho'|\rho) = \tilde{\mu}_b\check{\mathbf{g}}(\rho'|\rho)\\
&\check{\mathbf{g}}^{JH}(\rho'|\rho) = \check{\mathbf{g}}'(\rho'|\rho)\\
&\check{\mathbf{g}}^{MH}(\rho'|\rho) = \tilde{\sigma}_b\check{\mathbf{g}}(\rho'|\rho)\\
&\check{\mathbf{g}}^{ME}(\rho'|\rho) = \check{\mathbf{g}}'(\rho'|\rho)
\end{aligned}, \quad \check{\mathbf{g}}' = \nabla^* \times \check{\mathbf{g}}, \tag{5.A5}$$

where the function $\check{\mathbf{g}}(\rho'|\rho)$ is generated by the diagonal Green's tensor function $\hat{\mathbf{g}}(\rho'|\rho)$ for the homogeneous full space:

$$\check{\mathbf{g}} = \hat{\mathbf{g}} + k^{-2}\nabla^*(\nabla^* \cdot \hat{\mathbf{g}}), \qquad k^2 = \tilde{\mu}_b\tilde{\sigma}_b \neq 0. \tag{5.A6}$$

The Green's tensor function $\hat{\mathbf{g}}(\rho'|\rho)$ satisfies the tensor Helmholtz equation:

$$\nabla^{*2}\hat{\mathbf{g}} + k^2\hat{\mathbf{g}} = -\hat{\mathbf{I}}\frac{\delta(\rho-\rho')}{\rho}. \tag{5.A7}$$

In a cylindrical coordinate system it is determined by a differential matrix operator applied to the scalar function $g(\rho'|\rho)$:

$$\hat{\mathbf{g}} = \begin{pmatrix} -\frac{1}{\alpha^2}\frac{\partial^2}{\partial\rho\partial\rho'} & 0 & 0 \\ 0 & -\frac{1}{\alpha^2}\frac{\partial^2}{\partial\rho\partial\rho'} & 0 \\ 0 & 0 & 1 \end{pmatrix} g, \tag{5.A8}$$

$$g = \begin{cases} \mathbf{K}_0(\alpha\rho')\mathbf{I}_0(\alpha\rho), & \rho \le \rho' \\ \mathbf{I}_0(\alpha\rho')\mathbf{K}_0(\alpha\rho), & \rho > \rho' \end{cases}, \quad \alpha = \sqrt{\lambda^2 - k^2}, \quad \mathrm{Re}(\alpha) > 0. \tag{5.A9}$$

Here $\mathbf{I}_0$, $\mathbf{K}_0$ are the modified Bessel and MacDonald functions of the zero order.

Using Equations (5.A5), (5.A8) and (5.A9) and the similar notations as in Equations (5.5)–(5.9), we can calculate the spectra of the background fields,

$$\mathbf{e}^{\mathrm{b}} = \tilde{\mu}_{\mathrm{b}}\rho'\mathbf{i}_z \cdot \breve{\mathbf{g}}', \qquad \mathbf{h}^{\mathrm{b}} = k^2\rho'\mathbf{i}_z \cdot \breve{\mathbf{g}}, \tag{5.A10}$$

and the spectra of the Born approximations are

$$\mathbf{e}^{\mathrm{B}} = \mathbf{e}^{\mathrm{B}\sigma} + \mathbf{e}^{\mathrm{B}\mu} = \langle \tilde{\mu}_{\mathrm{b}}\breve{\mathbf{g}}\tilde{\sigma}_{\mathrm{a}}\mathbf{e}^{\mathrm{b}}\rangle_{V_\sigma(\rho)} + \langle \breve{\mathbf{g}}'\tilde{\mu}_{\mathrm{a}}\mathbf{h}^{\mathrm{b}}\rangle_{V_\mu(\rho)},$$

$$\mathbf{h}^{\mathrm{B}} = \mathbf{h}^{\mathrm{B}\sigma} + \mathbf{h}^{\mathrm{B}\mu} = \langle \breve{\mathbf{g}}'\tilde{\sigma}_{\mathrm{a}}\mathbf{e}^{\mathrm{b}}\rangle_{V_\sigma(\rho)} + \langle \tilde{\sigma}_{\mathrm{b}}\breve{\mathbf{g}}\tilde{\mu}_{\mathrm{a}}\mathbf{h}^{\mathrm{b}}\rangle_{V_\mu(\rho)}. \tag{5.A11}$$

The diagonal reflectivity tensors, $\hat{\lambda}^{\mathrm{e}}$, $\hat{\lambda}^{\mathrm{m}}$, are determined as the solutions of the linear system of equations obtained by the Fourier transform of Equations (5.22) in diagonal tensor form. In the case of an axial symmetrical model excited by the vertical magnetic dipole, the electrical reflectivity coefficient λ^{e} can be found from a scalar Equation (5.27). In particular, in the model with high-contrast conductivity and low-contrast magnetic susceptibility distributions (Figure 3), we can use the analytical expression for electrical reflectivity coefficient λ^{e}, based on Formula (5.33):

$$\lambda^{\mathrm{e}}(\rho) = \frac{e_\phi^{\mathrm{B}\sigma}(\rho)}{e_\phi^{\mathrm{b}}(\rho) - e_\phi^{\mathrm{B}\sigma}(\rho)}. \tag{5.A12}$$

Note that in this model we use the conventional Born approximation with respect to the magnetic anomaly and do not introduce the magnetic reflectivity coefficient λ^{m}. Using Equation (5.A12), we write

$$\left(1+\lambda^{\mathrm{e}}(\rho)\right)^{-1} = 1 - \frac{e_\phi^{B\sigma}(\rho)}{e_\phi^{\mathrm{b}}(\rho)}$$

$$= 1 - \tilde{\mu}_{\mathrm{b}}\tilde{\sigma}_{\mathrm{a}}\left(\int_{\rho_1}^{\rho} \mathbf{I}_1(\alpha\rho')\mathbf{K}_1(\alpha\rho')\rho'\,\mathrm{d}\rho' + \frac{\mathbf{I}_1(\alpha\rho)}{\mathbf{K}_1(\alpha\rho)}\int_{\rho}^{\rho_2} \mathbf{K}_1^2(\alpha\rho')\rho'\,\mathrm{d}\rho'\right), \tag{5.A13}$$

where $\alpha = \sqrt{\lambda^2 - k^2}$. The integrals in Formula (5.A13) can be evaluated in explicit form using the tabular integral (Gradstein and Ryzhik, 1994, p. 661, 5.54)

$$\int x\mathbf{U}_p(\alpha x)\mathbf{V}_p(\beta x)\,\mathrm{d}x = \frac{\beta x\mathbf{U}_p(\alpha x)\mathbf{V}_{p-1}(\beta x) - \alpha x\mathbf{U}_{p-1}(\alpha x)\mathbf{V}_p(\beta x)}{\alpha^2 - \beta^2},$$

where $\mathbf{U}_p(x)$and $\mathbf{V}_p(x)$ are arbitrary cylindrical functions of p order, and the interval $\rho' \in [\rho_1, \rho_2]$ is the thickness of the cylindrical layer with the anomalous conductivity $\tilde{\sigma}_{\mathrm{a}}$.

Finally, the vertical magnetic field H_z measured at the axis of symmetry at the point z_q is derived from the expressions

$$H_z = H_z^{\mathrm{b}} + \frac{1}{\pi}\int_0^\infty h_z(\lambda)\cos\lambda L\,\mathrm{d}\lambda,$$

$$H_z^{\mathrm{b}} = \frac{\mathrm{e}^{ikL}}{2\pi L^3}(1 - ikL),$$

$$h_z = \langle \mathbf{i}_z \cdot \hat{\mathbf{g}}'\tilde{\sigma}_{\mathrm{a}}(1+\lambda^{\mathrm{e}})\mathbf{e}^{\mathrm{b}}\rangle_{V_\sigma(\rho)} + \langle \mathbf{i}_z \cdot \tilde{\sigma}_{\mathrm{b}}\hat{\mathbf{g}}\tilde{\mu}_{\mathrm{a}}\mathbf{h}^{\mathrm{b}}\rangle_{V_\mu(\rho)}, \tag{5.A14}$$

where $L = |z_p - z_q|$ is the distance between the position of the transmitting magnetic dipole z_p and the position of the receiver z_q and k is the wave number of background homogeneous space.

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