CHAPTER 17

Maxwell's Equations and Numerical Electromagnetic Modeling in the Context of the Theory of Differential Forms

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Contents

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1. Introduction

The principal physical laws characterizing the behavior and interaction of electric and magnetic fields were unified in the comprehensive theory of electromagnetic fields by James Maxwell in his *Treatise on Electromagnetism*, first published in 1873 (Maxwell, 1954). The formulation of this theory represents one of the most important events in physics since Newton's time. In fact, Maxwell was the first to introduce the mathematical equations and physical laws which govern the electromagnetic field. Any effort to use electromagnetic fields to explore the earth must be firmly based on these physical laws and their mathematical consequences.

The fundamental system of electromagnetic (EM) field equations, Maxwell's equations, was developed by generalization of the basic laws of electromagnetism established in the first half of the 19th century. In the framework of classical theory, the EM field is described by the electric and magnetic vector fields, and Maxwell's equations represent a system of differential equations with respect to these vector fields.

During recent decades, an alternative approach was developed to the formulation of Maxwell's equations. This approach is based on the algebraic theory of differential forms and results in a very compact and symmetric system of differential form equations.

The differential forms were originally introduced in differential geometry to study the properties of the lines and surfaces in multidimensional mathematical spaces. However, it was realized not so long ago that these forms provide a very elegant and powerful tool to study the physical fields as well. We can treat the differential forms as another mathematical language which, similar to the vectorial language, can be used to describe the physical fields. In fact, in a four-dimensional space-time continuum, the differential forms can be treated as linear combinations of the differentials of the flux, the work, and/or the source of the vector fields. Therefore, Maxwell's equations for the differential forms contain the differentials of the flux and work of the electric and magnetic fields. This property of differential forms indicates that it is more suitable to consider the electric and/or magnetic flux and work as major characteristics of the EM field, instead of using the conventional vectorial representations. This approach seems to be quite reasonable from a physical point of view as well, because in physical experiments we, as a rule, measure the flux and the work (or voltage) of the electric and magnetic fields.

Moreover, the remarkable fact is that, based on the fundamental differential equations for the forms in a four-dimensional space, we can demonstrate that any pair of the arbitrary vector fields, $\mathbf{H}(\mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t)$, satisfies a system of differential equations which is similar to Maxwell's classical equations for electromagnetic fields! In other words, we can show that if a 2-form in the four-dimensional space-time E_4 is comprised of two arbitrary vector fields, $\mathbf{H}(\mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t)$, then these fields must automatically satisfy a system of Maxwell's equations.

The goal of this paper is to show that Maxwell's equations appear naturally from the basic equations of field theory for the differential forms. There are no other equations for a pair of nonstationary vector fields but equations of the Maxwell type. The basic laws of electromagnetism are actually imprinted in the fundamental differential relationships between the vector fields and differential forms. This new approach to the formulation and understanding of the basic properties of the laws of electromagnetism has the strong potential to stimulate future development in electromagnetic geophysics.

2. Differential Forms in Vector Field Theory

2.1. Concept of the differential form

In general, we can introduce the differential forms as expressions on which integration operates. There exist differential forms of different degrees depending on the dimensions of the domain of integration. In particular, a differential form of degree p, or a p-form, is an integrand of an integral over a domain of dimension p. We shall start our discussion by introducing the basic concept of differential forms in three-dimensional Euclidean space, where the degree of forms p varies from 0 to 3. A 0-form is a scalar function which is "integrated" over a region of zero dimension. In three-dimensional space the differential forms are closely related to the vector fields, and the algebraic and differential operations on the forms can be defined by conventional vector algebra and calculus, which makes it easier to understand for the reader familiar with conventional vector field theory.

In particular, a differential expression $\mathbf{B} \cdot \mathbf{dl}$, which is integrated over a curve, represents the elementary work, $\mathrm{d}W$, of the vector field $\mathbf{B}$ along an infinitesimally small vector element of curve $\mathbf{dl}$. It is called a *differential 1-form* $\underset{(1)}{\varphi}$:

$$\underset{(1)}{\varphi} = \mathbf{B} \cdot \mathbf{dl} = \mathrm{d}W. \tag{1}$$

A differential expression $\mathbf{B} \cdot \mathbf{ds}$, which is integrated over a surface, describes an elementary scalar flux, $\mathrm{d}F_{\mathbf{B}}^{\mathrm{d}s}$, of the vector field $\mathbf{B}$ through an infinitesimally small vector element of surface **ds**. It is called a *differential 2-form* $\underset{(2)}{\psi}$:

$$\underset{(2)}{\psi} = \mathbf{B} \cdot \mathbf{ds} = \mathrm{d}F_{\mathbf{B}}^{\mathbf{ds}}. \tag{2}$$

Finally, a differential expression $div\,\mathbf{B}\mathrm{d}v$, which is integrated over a volume, is equal to an elementary source, $\mathrm{d}Q$, of the vector field **B** within an infinitesimally small element of volume $\mathrm{d}v$. We call this expression a *differential 3-form* $\underset{(3)}{\theta}$:

$$\underset{(3)}{\theta} = q\mathrm{d}v = \mathrm{d}Q, \tag{3}$$

where

$$q = div\,\mathbf{B}.$$

It is known that the divergence of the vector field **B** can be treated as a source of this field. Thus, all three forms represent the scalar values of the work, flux, and source of the vector field, respectively.

2.2. Exterior (wedge) product of the differential forms

It can be shown that the conventional differential forms of vector calculus (the expressions that are integrated over a line, surface, or volume) are described by antisymmetric linear functions of one, two, or three vector arguments (Zhdanov, 2009). These linear functions (differential forms) represent new mathematical objects which are very useful in a description of the electromagnetic field equations.

The differential forms provide the most natural and elegant mathematical tool for a description of electromagnetic fields (Lindell, 2004; Zhdanov, 2009). In order to be able to apply these functions to electromagnetic theory, we should introduce the mathematical rules of operation on the differential forms, which define the *algebra* of the differential forms.

The simplest operation is addition. It is obvious that the addition of two antisymmetric linear forms is determined as a conventional summation of two functions. This operation satisfies the traditional commutative and associative laws, and also the distributive laws with respect to multiplication by a scalar.

The multiplication of the differential forms requires the introduction of a special algebraic operation, an exterior product, which we will discuss in detail below.

In the case of vectors, we can use different multiplication operations, e.g., dot and cross products of the vectors. In principle, it is easy to consider a product, f, of two forms, φ and ψ, as a product of two linear functions of vector arguments. For example, if $\varphi = \varphi(\mathbf{dl})$ and $\psi = \psi(\mathbf{da}, \mathbf{db})$, we have

$$f = f(\mathbf{dl}, \mathbf{da}, \mathbf{db}) = \varphi(\mathbf{dl})\psi(\mathbf{da}, \mathbf{db}), \tag{4}$$

where multiplication on the right-hand side of Eq. (4) is conducted in the conventional way as a product of two scalar values, ϕ and ψ. The only problem with this definition is that the product of the two antisymmetric linear functions, ϕ and ψ is no longer an antisymmetric function! Indeed, one can see that

$$f(\mathbf{dl}, \mathbf{da}, \mathbf{db}) \neq -f(\mathbf{da}, \mathbf{dl}, \mathbf{db}).$$

In order to keep the result of multiplication within the class of antisymmetric linear functions, we should apply an antisymmetrization operation to the conventional product (4). As a result, we arrive at the following multiplication operation, which is called the exterior (or wedge) product, denoted by the symbol $\wedge$, and defined as

$$\begin{aligned}\Omega &= \varphi \wedge \psi \\ &= \{\varphi(\mathbf{dl})\,\psi(\mathbf{da}, \mathbf{db}) + \varphi(\mathbf{da})\,\psi(\mathbf{db}, \mathbf{dl}) + \varphi(\mathbf{db})\,\psi(\mathbf{dl}, \mathbf{da}) \\ &\quad - \varphi(\mathbf{dl})\,\psi(\mathbf{db}, \mathbf{da}) - \varphi(\mathbf{da})\,\psi(\mathbf{dl}, \mathbf{db}) - \varphi(\mathbf{db})\,\psi(\mathbf{da}, \mathbf{dl})\}.\end{aligned} \tag{5}$$

It is easy to verify that the new linear function, $\Omega = \Omega(\mathbf{dl}, \mathbf{da}, \mathbf{db})$, is indeed an antisymmetric function.

In the case of the product of two 1-forms, $\phi(\mathbf{da})$ and $\chi(\mathbf{db})$, the exterior product is given by the following formula:

$$\Phi(\mathbf{da}, \mathbf{db}) = \varphi(\mathbf{da}) \wedge \chi(\mathbf{db}) = \varphi(\mathbf{da})\,\chi(\mathbf{db}) - \varphi(\mathbf{db})\,\chi(\mathbf{da}), \tag{6}$$

which is again an antisymmetric function:

$$\Phi(\mathbf{da}, \mathbf{db}) = -[\varphi(\mathbf{db})\,\chi(\mathbf{da}) - \varphi(\mathbf{da})\,\chi(\mathbf{db})] = -\Phi(\mathbf{db}, \mathbf{da}). \tag{7}$$

In summary, we can see that the addition and exterior multiplication operations of 1-forms satisfy the following laws:

(1) the commutative and associative laws of addition:

$$\varphi + \chi = \chi + \varphi \quad \text{and} \quad \varphi + (\chi + \eta) = (\varphi + \chi) + \eta; \tag{8}$$

(2) the anticommutative law of exterior multiplication:

$$\varphi \wedge \chi = -\chi \wedge \varphi; \tag{9}$$

(3) the distributive law of multiplication over addition:

$$\begin{aligned} &\varphi \wedge (\chi + \eta) = \varphi \wedge \chi + \varphi \wedge \eta \quad \text{and} \\ &(\varphi + \chi) \wedge \eta = \varphi \wedge \eta + \chi \wedge \eta. \end{aligned} \tag{10}$$

It comes immediately from the anticommutative law (9) that

$$\varphi \wedge \varphi = 0. \tag{11}$$

We can see now that the exterior product of two 1-forms generates a 2-form, while the exterior product of three 1-forms produces a 3-form.

2.3. Canonical representations of the differential forms in three-dimensional Euclidean space

The important fact of the differential form theory is that any 2-form and any 3-form can be expressed as the exterior products of two 1-forms and three 1-forms respectively. These representations are called the *canonical representations* for the differential forms (Zhdanov, 2009). The following table presents a summary of these canonical representations for differential forms in three-dimensional Euclidean space E_3:

$$\text{0-form}: \underset{(0)}{\varphi} = f, \tag{12}$$

$$\text{1-form}: \underset{(1)}{\varphi} = \sum_{\alpha=x,y,z} \varphi_\alpha \mathrm{d}\alpha = \varphi_x \mathrm{d}x + \varphi_y \mathrm{d}y + \varphi_z \mathrm{d}z = \boldsymbol{\varphi} \cdot \mathrm{d}\mathbf{r}, \tag{13}$$

$$\begin{aligned} \text{2-form}: \underset{(2)}{\psi} &= \sum_{\alpha,\beta=x,y,z} \boldsymbol{\psi}_{\alpha\beta} \mathrm{d}\alpha \wedge \mathrm{d}\beta = \psi_{yz} \mathrm{d}y \wedge \mathrm{d}z \\ &+ \psi_{zx} \mathrm{d}z \wedge \mathrm{d}x + \psi_{xy} \mathrm{d}x \wedge \mathrm{d}y = \boldsymbol{\psi} \cdot \mathrm{d}\boldsymbol{\Sigma}, \end{aligned} \tag{14}$$

$$\text{3-form}: \underset{(3)}{\theta} = \theta \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \theta \mathrm{d}v. \tag{15}$$

In these last formulas we used the following notations:

$$\begin{aligned} &\boldsymbol{\varphi} = \left(\varphi_x, \varphi_y, \varphi_z\right), \quad \mathrm{d}\mathbf{r} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z), \\ &\boldsymbol{\psi} = \psi_{yz}\mathbf{d}_x + \psi_{zx}\mathbf{d}_y + \psi_{xy}\mathbf{d}_z \\ &\mathrm{d}\boldsymbol{\Sigma} = \mathrm{d}s_x\mathbf{d}_x + \mathrm{d}s_y\mathbf{d}_y + \mathrm{d}s_z\mathbf{d}_z, \end{aligned}$$

where $\mathrm{d}s_x$, $\mathrm{d}s_y$, and $\mathrm{d}s_z$ are the combinations of the exterior products of differentials:

$$\mathrm{d}s_x = \mathrm{d}y \wedge \mathrm{d}z, \quad \mathrm{d}s_y = \mathrm{d}z \wedge \mathrm{d}x, \quad \mathrm{d}s_z = \mathrm{d}x \wedge \mathrm{d}y.$$

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2.4. The exterior derivative operation

The calculus of differential forms is based on a special differential operation called the *exterior derivative*. This operation can be treated as a generalization of conventional vector differential operations. In fact, all three different vector differential operations (gradient, divergence, and curl) can be represented as a single exterior differential operator.

0-forms

In the case of the simplest 0-form described by a function $f(\mathbf{r})$, the exterior differential is equivalent to the full differential of the function:

$$df = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = grad\, f(\mathbf{r}) \cdot \mathbf{dr}. \tag{16}$$

We can see that in this case the exterior differential operator $\mathbf{d}$ can be treated as a counterpart of the vector *del* operator ∇. We can introduce a symbolic differential 1-form $\mathbf{d}_{E_3}$ as follows:

$$\mathbf{d}_{E_3} = \mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y} + \mathrm{d}z\frac{\partial}{\partial z} = \mathbf{dr} \cdot \nabla. \tag{17}$$

Then Eq. (16) can be written as

$$df = \mathrm{d}E_3 f = \mathbf{dr} \cdot \nabla f = \nabla f \cdot \mathbf{dr}.$$

Therefore, the exterior derivative of the 0-form is equivalent to the gradient of the corresponding scalar field.

1-forms

The exterior differential of the 1-form field $\phi(\mathbf{r})$ is calculated as an exterior product of the differential operator 1-form $\mathbf{d}_{E_3}$ and a given 1-form ϕ:

$$\begin{aligned} \mathrm{d}\varphi &= \mathbf{d}_{E_3} \wedge \varphi = \mathbf{d}_{E_3} \wedge (\varphi \cdot \mathbf{dr}) \\ &= curl\, \varphi \cdot \mathbf{d\Sigma} = [\nabla \times \varphi] \cdot \mathbf{d\Sigma} \end{aligned} \tag{18}$$

where we took into account representations (13) and (14) for the differential forms. Therefore, the exterior derivative of the 1-form is equivalent to the curl of the corresponding vector field.

2-forms

The exterior differential of the 2-form field $\psi(\mathbf{r})$ is equal to:

$$\begin{aligned} \mathrm{d}\boldsymbol{\psi} &= \mathbf{d}_{E_3} \wedge \psi = \mathbf{d}_{E_3} \wedge (\boldsymbol{\psi} \cdot \mathbf{d\Sigma}) \\ &= (div\, \boldsymbol{\psi})\, \mathrm{d}v = (\nabla \cdot \boldsymbol{\psi})\, \mathrm{d}v. \end{aligned} \tag{19}$$

According to the canonical representation (14), every 2-from can be described by a vector field $\boldsymbol{\psi}$. Therefore, the exterior derivative of the 2-form is equivalent to the divergence of the corresponding vector field.

3-forms

The exterior differential of the 3-form $\underset{(3)}{\theta} = \theta\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ can be calculated as follows:

$$\mathrm{d}_{E_3} \wedge \underset{(3)}{\theta} = \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

This last expression is equal to zero according to the anticommutative law of the wedge product:

$$\mathrm{d}_{E_3} \wedge \underset{(3)}{\theta} = \left(\frac{\partial \theta}{\partial x}\mathrm{d}x + \frac{\partial \theta}{\partial y}\mathrm{d}y + \frac{\partial \theta}{\partial z}\mathrm{d}z \right) \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 0. \quad (20)$$

Thus we can see that, the exterior differential operation in three-dimensional space corresponds to either the gradient, curl, or divergence of conventional vector calculus:

$$\text{0-forms}: \ \mathrm{d}_{E_3} \underset{(0)}{\varphi} = \mathrm{d}_{E_3} f = \mathit{grad}\, f(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \nabla f(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}, \quad (21)$$

$$\text{1-forms}: \ \mathrm{d}_{E_3} \underset{(1)}{\varphi} = \mathrm{d}_{E_3} \wedge (\varphi \cdot \mathrm{d}\mathbf{r}) = \mathit{curl}\, \varphi \cdot \mathrm{d}\boldsymbol{\Sigma} = [\nabla \times \varphi] \cdot \mathrm{d}\boldsymbol{\Sigma}, \quad (22)$$

$$\text{2-forms}: \ \mathrm{d}_{E_3} \underset{(2)}{\varphi} = \mathrm{d}_{E_3} \wedge (\boldsymbol{\psi} \cdot \mathrm{d}\boldsymbol{\Sigma}) = (\mathit{div}\, \boldsymbol{\psi})\, \mathrm{d}v = (\nabla \cdot \boldsymbol{\psi})\, \mathrm{d}v, \quad (23)$$

$$\text{3-forms}: \ \mathrm{d}_{E_3} \underset{(3)}{\varphi} = \mathrm{d}_{E_3} \wedge (\theta\, \mathrm{d}v) = 0. \quad (24)$$

The beauty of the exterior differentiation operator is that it includes all varieties of vector differential operations. In addition, this operator allows us to extend naturally the differentiation operation for multidimensional spaces.

3. Nonstationary Field Equations and Differential Forms

Geophysical methods are based on studying the propagation of the different physical fields within the earth's interior. Two of the most widely used fields in geophysics are seismic and electromagnetic fields, which are typical nonstationary fields. These fields can be represented as vector fields in four-dimensional Euclidean space-time E_4. In this paper we will derive the general system of differential equations for nonstationary vector fields. We will demonstrate that these equations are nothing else but Maxwell-type equations.

3.1. Nonstationary vector fields and differential forms in four-dimensional space E_4

The most effective way of presenting the theory of nonstationary (time-dependent) fields is based on the theory of differential forms. We introduce the four-dimensional Euclidean space E_4, which has three conventional spatial coordinates, $x_1 = x$, $x_2 = y$ and $x_3 = z$, and the fourth coordinate, x_4, equal to time: $x_4 = t$.

Any scalar, U, or vector, $\mathbf{A}$, functions of the space coordinates (x_1, x_2, x_3) and time coordinate t can be treated as functions defined in the four-dimensional space E_4. We can also introduce arbitrary vector fields $\mathbf{H}(\mathbf{r}, t)$, $\mathbf{D}(\mathbf{r}, t)$, and $\mathbf{j}(\mathbf{r}, t)$, all of which are nonstationary (time-dependent) vector functions in three-dimensional space, and a scalar function $q(\mathbf{r}, t)$. The remarkable fact is that any pair of nonstationary fields, $\mathbf{H}(\mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t)$, satisfies a set of differential equations which have exactly the same structure as Maxwell's equation of electromagnetic theory! We will derive these equations using the differential form theory.

One can show that, using the vector and scalar fields $\mathbf{A}, \mathbf{H}, \mathbf{D}, \mathbf{j}$ and q, we can define differential forms of five different orders in the four-dimensional space E_4. These forms can be expressed using the three-dimensional vector notations as follows (Zhdanov, 2009):

$$\text{0-forms}: \underset{(0)}{\Omega} = U, \tag{25}$$

$$\text{1-forms}: \underset{(1)}{\Omega} = \alpha = \mathbf{A} \cdot \mathrm{d}\mathbf{r} - U \mathrm{d}t, \tag{26}$$

$$\text{2-forms}: \underset{(2)}{\Omega} = \psi = \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma} - (\mathbf{H} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}t, \tag{27}$$

$$\text{3-forms}: \underset{(3)}{\Omega} = \gamma = q \mathrm{d}v - (\mathbf{j} \cdot \mathrm{d}\boldsymbol{\Sigma}) \wedge \mathrm{d}t, \tag{28}$$

$$\text{4-forms}: \underset{(4)}{\Omega} = \theta = q \mathrm{d}v \wedge \mathrm{d}t. \tag{29}$$

3.2. Differential form equations

It is known that any p-form in four-dimensional space E_4 can be split into two terms which are called its *spatial*, Ω_s, *and temporal*, Ω_τ, *components* (Lindell, 2004; Fecko, 2006):

$$\underset{(p)}{\Omega} = \Omega_s + \Omega_t \wedge \mathrm{d}t.$$

Note that the time coordinate and the spatial coordinates are mutually orthogonal in E_4. Therefore, any differential form equation in the space E_4, e.g.,

$$\underset{(p)}{\Omega} = 0,$$

can be split into separate equations for the spatial and temporal components:

$$\Omega_s = 0 \quad \text{and} \quad \Omega_\tau = 0.$$

We will summarize below the basic differential equations for the differential forms in the four-dimensional space E_4.

3.3. Exterior derivative of a scalar field and a generalized source 1-form

We begin with the exterior derivative of the 0-form:

$$\mathrm{d}\underset{(0)}{\Omega} = \mathrm{d}U = grad\, U \cdot \mathbf{dr} + \frac{\partial}{\partial t} U \mathrm{d}t = \underset{(1)}{g}. \tag{30}$$

According to Eq. (26), the 1-form $\underset{(1)}{g}$ can be written as:

$$\underset{(1)}{g} = g_{1s} + g_{1\tau}\mathrm{d}t = \mathbf{g}_1 \cdot \mathbf{dr} + g_{1\tau}\mathrm{d}t. \tag{31}$$

Splitting Eq. (30) into its spatial and temporal parts, we find:

$$g_{1s} = \mathbf{g}_1 \cdot \mathbf{dr} = grad\, U \cdot \mathbf{dr} \quad \text{and} \quad g_{1\tau} = \frac{\partial}{\partial t} U. \tag{32}$$

The 1-form $\underset{(1)}{g}$ is called *a generalized source form of the 0-form field* $\underset{(0)}{\Omega}$. Its spatial component is equal to the work of the gradient of a scalar field U, along a vector element $\mathbf{dr}$, while its temporal component is equal to the time derivative of the scalar field U.

3.4. Exterior derivative of a four-potential and a generalized source 2-form

It can be demonstrated that the exterior derivative of the 1-form α (so-called four-potential) is given by the following expression (Zhdanov, 2009):

$$\mathrm{d}\underset{(1)}{\Omega} = \mathrm{d} \wedge \alpha = curl\,\mathbf{A} \cdot \mathrm{d}\boldsymbol{\Sigma} - \left(grad\, U + \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \mathbf{dr} \wedge \mathrm{d}t = \underset{(2)}{g}, \tag{33}$$

where $\underset{(2)}{g}$ is a generalized source 2-form:

$$\underset{(2)}{g} = g_{2s} + g_{2\tau}\mathrm{d}t = \mathbf{j}_A \cdot \mathrm{d}\boldsymbol{\Sigma} - \left(grad\, U + \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \mathrm{d}\mathbf{r} \wedge \mathrm{d}t, \tag{34}$$

and $\mathbf{j}_A = curl\, \mathbf{A}$.

Splitting Eq. (34) into its spatial and temporal equations, we find:

$$g_{2s} = \mathbf{j}_A \cdot \mathrm{d}\boldsymbol{\Sigma} \quad \text{and} \quad g_{2\tau} = -\left(grad\, U + \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \mathrm{d}\mathbf{r}. \tag{35}$$

3.5. Exterior derivative of a 2-form and a four-current

Similarly, we can find the exterior derivative of the 2-form:

$$\mathrm{d}\underset{(2)}{\Omega} = \mathrm{d} \wedge \psi = (div\, \mathbf{D})\, \mathrm{d}v + \left(\frac{\partial}{\partial t}\mathbf{D} - curl\, \mathbf{H} \right) \cdot \mathrm{d}\boldsymbol{\Sigma} \wedge \mathrm{d}t = \underset{(3)}{g}, \tag{36}$$

where $\underset{(3)}{g}$ is a generalized source 3-form or so-called four-current γ^{ψ}:

$$\underset{(3)}{g} = g_{3s} + g_{3\tau}\mathrm{d}t = q^{\psi}\mathrm{d}v - \left(\mathbf{j}^{\psi} \cdot \mathrm{d}\boldsymbol{\Sigma}\right) \wedge \mathrm{d}t = \gamma^{\psi}, \tag{37}$$

and

$$\mathbf{j}^{\psi} = curl\, \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} \quad \text{and} \quad q^{\psi} = div\, \mathbf{D}. \tag{38}$$

From Eqs (37) and (38) we find the spatial and temporal components of the generalized source 3-form (four-current γ^{ψ}):

$$g_{3s} = (div\, \mathbf{D})\, \mathrm{d}v, \quad g_{3\tau} = \left(\frac{\partial}{\partial t}\mathbf{D} - curl\, \mathbf{H} \right) \cdot \mathrm{d}\boldsymbol{\Sigma}. \tag{39}$$

3.6. Exterior derivatives of a 3-form and a 4-form

Finally, the exterior derivative of a 3-form is equal to:

$$\mathrm{d}\underset{(3)}{\Omega} = \mathrm{d} \wedge \gamma = -\left(div\, \mathbf{j} + \frac{\partial}{\partial t} q \right) \mathrm{d}v \wedge \mathrm{d}t = \underset{(4)}{g}. \tag{40}$$

The generalized source 4-form $\underset{(4)}{g}$, according to Eq. (29), can be written as:

$$\underset{(4)}{g} = g_{3\tau}\mathrm{d}t = q^{\gamma}\mathrm{d}v \wedge \mathrm{d}t, \tag{41}$$

where

$$q^{\gamma} = -div\,\mathbf{j} - \frac{\partial}{\partial t}q.$$

The exterior derivative of the 4-form θ in a four-dimensional space is always equal to zero:

$$\mathrm{d} \wedge \theta = 0.$$

We can summarize all these results as:

$$\text{0-forms}: \ \mathrm{d}U = grad\,U \cdot \mathrm{d}\mathbf{r} + \frac{\partial}{\partial t}U\mathrm{d}t, \tag{42}$$

$$\text{1-forms}: \ \mathrm{d} \wedge \alpha = curl\,\mathbf{A} \cdot \mathrm{d}\mathbf{\Sigma} - \left(grad\,U + \frac{\partial \mathbf{A}}{\partial t}\right) \cdot \mathrm{d}\mathbf{r} \wedge \mathrm{d}t, \tag{43}$$

$$\text{2-forms}: \ \mathrm{d} \wedge \psi = (div\,\mathbf{D})\,\mathrm{d}v + \left(\frac{\partial}{\partial t}\mathbf{D} - curl\,\mathbf{H}\right) \cdot \mathrm{d}\mathbf{\Sigma} \wedge \mathrm{d}t, \tag{44}$$

$$\text{3-forms}: \ \mathrm{d} \wedge \gamma = -\left(div\,\mathbf{j} + \frac{\partial}{\partial \tau}q\right)\mathrm{d}v \wedge \mathrm{d}t. \tag{45}$$

$$\text{4-forms}: \ \mathrm{d} \wedge \theta = 0. \tag{46}$$

4. Ampere-type Differential Forms and a Continuity Equation

According to Eqs (36) and (37), we have the following differential equation for any 2-form $\psi = \mathbf{D} \cdot \mathrm{d}\mathbf{\Sigma} - (\mathbf{H} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}\tau$:

$$\mathrm{d} \wedge \psi = \gamma^{\psi}, \tag{47}$$

where the corresponding four-current 3-form γ^{ψ} is equal to:

$$\gamma^{\psi} = q^{\psi}\mathrm{d}v - \left(\mathbf{j}^{\psi} \cdot \mathrm{d}\mathbf{\Sigma}\right) \wedge \mathrm{d}t, \tag{48}$$

and

$$j^{\psi} = curl\,\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} \quad \text{and} \quad q^{\psi} = div\,\mathbf{D}. \tag{49}$$

Note that, according to the basic property of the exterior derivative operator, the double application of the external differential is identically equal to zero. Therefore, the four-current γ^{ψ} satisfies the following equation:

$$\mathrm{d} \wedge \gamma^{\psi} = \mathrm{d} \wedge (\mathrm{d} \wedge \psi) = 0. \tag{50}$$

This last equation can be written, according to formula (40), as:

$$div\, \mathbf{j}^{\psi} + \frac{\partial}{\partial t} q^{\psi} = 0. \tag{51}$$

We can see that Eq. (49) represents Maxwell's equations exactly if vector fields $\mathbf{D}$ and $\mathbf{H}$ are treated as the electric and magnetic fields, respectively. Correspondingly, Eq. (51) represents a conservation law for the four-current and is called *the continuity equation* because it has the form of the continuity equation of electromagnetic theory.

Equation (47) is called *a fundamental differential equation for 2-forms*, because any 2-form in the four-dimensional space E_4 must satisfy this equation. At the same time, Eq. (49) are nothing else but Maxwell's first and fourth equations for the electric field $\mathbf{D}$ and magnetic field $\mathbf{H}$, which describe Ampere's law of electromagnetism with Maxwell's displacement current $\partial \mathbf{D}/\partial t$. Thus, Maxwell's equations appear naturally from the general theory of differential forms.

The 2-form ψ, which satisfies the fundamental equation (47), is called *an Ampere-type differential form*. We should note, however, that actually every 2-form in the four-dimensional Euclidean space E_4 is an Ampere-type form. Its spatial component, ψ_s, is called an *electric induction* 2-form D, while its temporal component, ψ_τ, is called a magnetic 1-form H:

$$D = \psi_s = \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma}, \quad H = \psi_\tau = \mathbf{H} \cdot \mathrm{d}\mathbf{r}. \tag{52}$$

Thus, the Ampere-type differential form can be written as:

$$\psi = D - H \wedge \mathrm{d}t. \tag{53}$$

5. Faraday-Type Differential Forms and Four-Potential

Let us consider now a special class of 2-forms, which satisfies Eq. (47) with a zero right-hand part:

$$\mathrm{d} \wedge \phi = 0. \tag{54}$$

In other words, we assume now that the corresponding four-current is equal to zero, $\gamma_\phi = 0$, and the 2-form ϕ is an exact form.

In this case, according to de Rham's theorem (Lindell, 2004), there exists a 1-form (four-potential) α,

$$\alpha = \mathbf{A} \cdot \mathrm{d}\mathbf{r} - U\mathrm{d}t,$$

such that

$$\phi(\mathbf{r}) = \mathrm{d} \wedge \alpha(\mathbf{r}). \tag{55}$$

Equation (55) can be written in an equivalent way as:

$$\begin{aligned}\phi &= \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma} + (\mathbf{E} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}t = \mathrm{d} \wedge \alpha(\mathbf{r}) \\ &= \mathit{curl}\,\mathbf{A} \cdot \mathrm{d}\boldsymbol{\Sigma} - \left(\mathit{grad}\,U + \frac{\partial \mathbf{A}}{\partial t}\right) \cdot \mathrm{d}r \wedge \mathrm{d}t,\end{aligned} \tag{56}$$

where $\mathbf{B}$ and $\mathbf{E}$ are some conventional nonstationary (time-dependent) vector functions in three-dimensional space. These functions, according to formulas (54) and (38), satisfy the following equations:

$$\mathit{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \mathit{div}\,\mathbf{B} = 0. \tag{57}$$

Remarkably, Eqs (57) are nothing else but Maxwell's second and third equations for electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, which describe the Faraday law of electromagnetic induction and the fact of an absence of magnetic charges. That is why the 2-form ϕ, which satisfies Eq. (54), is called *a Faraday-type differential form*. Its spatial component, ϕ_s, is called a *magnetic induction* 2-form $\mathbf{B}$, while its temporal component, ϕ_τ, is called an electric 1-form E:

$$B = \phi_s = \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma}, \quad E = \phi_\tau = \mathbf{E} \cdot \mathrm{d}\mathbf{r}. \tag{58}$$

Thus, the Faraday-type differential form can be written as:

$$\psi = B + E \wedge \mathrm{d}t. \tag{59}$$

Equation (56) shows that every Faraday-type form can be expressed by the corresponding four-potential α. Splitting Eq. (56) into its spatial and temporal equations, we find that:

$$B = \mathbf{B} \cdot d\mathbf{\Sigma} = curl\,\mathbf{A} \cdot d\mathbf{\Sigma}, \quad E = \mathbf{E} \cdot d\mathbf{r} = -\left(grad\,U + \frac{\partial \mathbf{A}}{\partial t}\right) \cdot d\mathbf{r}. \quad (60)$$

Finally we arrive at the conventional representation for the vector fields **B** and **E** by the vector **A** and scalar U potentials:

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\left(grad\,U + \frac{\partial \mathbf{A}}{\partial t}\right). \quad (61)$$

6. MAXWELL'S EQUATIONS

6.1. Basic equations in the theory of electromagnetic fields

Maxwell's equations consist of the two vector equations and two scalar equations shown below:

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{c}, \quad (62)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (63)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (64)$$

$$\nabla \cdot \mathbf{D} = q. \quad (65)$$

Here, **H** and **B** are the vector magnetic and induction fields, respectively; **E** and **D** are the vector electric and displacement fields, respectively; q is the electrical charge density; **j** is the conduction current density; and **c** is the total current density (the sum of conduction and displacement currents). The pairs of fields, **E** and **D**, **H** and **B**, are related by the following expressions, known as the constitutive equations:

$$\mathbf{D} = \varepsilon\,\mathbf{E}, \quad (66)$$

$$\mathbf{B} = \mu\,\mathbf{H}, \quad (67)$$

where ε and μ are dielectric and magnetic permeabilities of the medium, respectively.

Maxwell's equations were introduced by generalization of the basic laws of electromagnetism established in the first half of the 19th century. It is interesting to note that in fact all these equations can be derived directly from the basic differential equations of field theory, formulated above. Indeed, let us introduce two electromagnetic differential 2-forms, M and F, according to the following formulae:

$$M = \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma} - (\mathbf{H} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}t = D - H \wedge \mathrm{d}t, \tag{68}$$

$$F = \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma} + (\mathbf{E} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}t = B + E \wedge \mathrm{d}t, \tag{69}$$

where $D = \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma}$, $H = \mathbf{H} \cdot \mathrm{d}\mathbf{r}$, $B = \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma}$, and $E = \mathbf{E} \cdot \mathrm{d}\mathbf{r}$.

Following Misner et al. (1973) and Deschamps (1981), we will call these forms Maxwell's field, M, and force field, F, respectively. Using the basic properties of differential 2-forms, discussed above, we can write the following differential equations for these forms:

$$\mathrm{d} \wedge M = \gamma^e \tag{70}$$

$$\mathrm{d} \wedge F = \gamma^m, \tag{71}$$

where the corresponding electric, γ^e, and magnetic, γ^m, four-currents are equal to:

$$\gamma^e = q\mathrm{d}v - (\mathbf{j} \cdot \mathrm{d}\boldsymbol{\Sigma}) \wedge \mathrm{d}t, \tag{72}$$

$$\gamma^m = q^m\mathrm{d}v - \left(\mathbf{j}^m \cdot \mathrm{d}\boldsymbol{\Sigma}\right) \wedge \mathrm{d}t. \tag{73}$$

Here, the functions q^m and $\mathbf{j}^m$ are the magnetic charge density and the magnetic current density, respectively.

According to formulae (47) and (49), from the differential equation (70) for Maxwell's field M we obtain immediately Maxwell's first and fourth equations (62) and (65):

$$\mathit{curl}\,\mathbf{H} = \mathbf{j} + \frac{\partial}{\partial t}\mathbf{D} \quad \text{and} \quad \mathit{div}\,\mathbf{D} = q. \tag{74}$$

Taking into account that the external differential of the four-current γ^e, according to (70), is equal to zero,

$$\mathrm{d} \wedge \gamma^e = \mathrm{d} \wedge \mathrm{d} \wedge M = 0,$$

and considering formula (40), we arrive at the *continuity equation* for electric current density $\mathbf{j}$ and the charge density q:

$$\nabla \cdot \mathbf{j} = -\frac{\partial q}{\partial t}. \tag{75}$$

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In a similar way, from the differential equation (71) for the force field F we obtain immediately a generalization of Maxwell's second and third equations (63) and (64), which allows the existence of the magnetic charges:

$$curl\,\mathbf{E} = -\mathbf{j}^m - \frac{\partial}{\partial t}\mathbf{B} \quad \text{and} \quad div\,\mathbf{B} = q^m. \tag{76}$$

Note that the magnetic four-current γ^m satisfies the same differential equation as the electric four-current:

$$\mathrm{d} \wedge \gamma^m = \mathrm{d} \wedge \mathrm{d} \wedge F = 0.$$

Therefore, the magnetic charges and current, in general cases, are related by the continuity equation as well:

$$\nabla \cdot \mathbf{j} = -\frac{\partial q^m}{\partial t}. \tag{77}$$

Introducing the magnetic charges makes Maxwell's equation symmetrical. However, in the real world we do not observe the magnetic charges, which results in a Faraday-type equation for the force field:

$$\mathrm{d} \wedge F = 0. \tag{78}$$

This equation, written in vectorial notation, brings us to Maxwell's original second and third equations (63) and (64):

$$curl\,\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \quad \text{and} \quad div\,\mathbf{B} = 0.$$

Thus, the whole system of Maxwell's equations automatically appears from the general theory of nonstationary fields. This remarkable fact demonstrates the power of mathematical theory. We can see now that the basic laws of electromagnetism are actually hidden in the fundamental differential relationships between the vector fields and the differential forms.

In summary, we can represent now the full system of Maxwell's equations in a very compact and elegant form as follows:

$$\mathrm{d} \wedge M = \gamma^e, \tag{79}$$

$$\mathrm{d} \wedge F = 0, \tag{80}$$

where

$$M = \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma} - (\mathbf{H} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}t = D - H \wedge \mathrm{d}t, \tag{81}$$

and

$$F = \mathbf{B} \cdot d\boldsymbol{\Sigma} + (\mathbf{E} \cdot d\mathbf{r}) \wedge dt = B + E \wedge dt. \quad (82)$$

It was demonstrated above that any Faraday-type form can be expressed by the corresponding four-potential α. Taking into account that the force field F is a Faraday-type form, we can use Eq. (60), according to which,

$$B = \mathbf{B} \cdot d\boldsymbol{\Sigma} = curl\,\mathbf{A} \cdot d\boldsymbol{\Sigma}, \quad E = \mathbf{E} \cdot d\mathbf{r} = -\left(grad\,U + \frac{\partial \mathbf{A}}{\partial t}\right) \cdot d\mathbf{r}. \quad (83)$$

From the last formula we obtain a classical representation of the magnetic and electric fields using *electrodynamic potentials*, **A**and U:

$$\mathbf{B} = curl\,\mathbf{A} \quad \text{and} \quad \mathbf{E} = -\left(grad\,U + \frac{\partial \mathbf{A}}{\partial t}\right). \quad (84)$$

7. Integral Formulations of the Differential Form Equations for Maxwell's Field and Force Field

Equations (79) and (80) represent a differential (localized) formulation of the laws of electromagnetism. In applications, it is useful to have the integral formulations of the same laws. For example, we will demonstrate below that the integral formulations are useful in numerical modeling of EM fields.

7.1. Faraday's electromagnetic induction law

Let us consider Maxwell's equation for the force field first:

$$d \wedge F = 0. \quad (85)$$

We can integrate this equation over a geometrical element C_p ($p = 1, 2, 3, 4$) from the four-dimensional Euclidean space E_4:

$$\int_{C_p} d \wedge F = 0, \quad (86)$$

where the geometrical element C_p can be treated as a p-dimensional domain in a multidimensional space. For example, we can consider a geometrical element C_3 in a form of the "cylinder" over some surface S in three-dimensional Euclidean space E_3, with the conventional spatial coordinates,

$x_1 = x$, $x_2 = y$, and $x_3 = z$, and a time interval $T = (t_0 \leq t \leq t_0 + \Delta t)$. Using standard mathematical notations, we can present the geometrical element C_3 in the form:

$$C_3 = T \times S.$$

According to a general Stokes's theorem (Fecko, 2006), the integral of the exact p-form $\underset{(p)}{\Omega} = \mathrm{d} \wedge \underset{(p-1)}{\Omega}$ over a geometrical element C_p is equal to the integral of the $(p$-1$)$-form $\underset{(p-1)}{\Omega}$ over the boundary, ∂C_p, of the geometrical element C_p:

$$\int_{C_p} \underset{(p)}{\Omega} = \int_{C_p} \mathrm{d} \wedge \underset{(p-1)}{\Omega} = \int_{\partial C_p} \underset{(p-1)}{\Omega}, \quad p = 1, 2, 3, 4. \tag{87}$$

According to Maxwell's equation (85), the force field F is an exact 2-form. Integrating both sides of Eq. (85) over the geometrical element $C_3 = T \times S$ and taking into account the general Stokes's theorem (87), we obtain

$$\int_{C_3} \mathrm{d} \wedge F = \int_{\partial C_3} F = 0. \tag{88}$$

Substituting expression (85) for the force field in Eq. (88), we have:

$$\begin{aligned} \int_{\partial C_3} F &= \int_{\partial C_3} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma} + \int_{\partial C_3} (\mathbf{E} \cdot \mathrm{d}\mathbf{r}) \wedge \mathrm{d}t \\ &= \int_{\partial C_3} B + \int_{\partial C_3} E \wedge \mathrm{d}t = 0. \end{aligned} \tag{89}$$

A simple geometrical consideration shows that

$$\partial C_3 = \partial T \times S - T \times \partial S = [t_0 + \Delta t] \times S - [t_0] \times S - T \times \partial S. \tag{90}$$

Taking into account the geometrical structure (90) of the boundary ∂C_3, we can calculate the integrals in Eq. (89):

$$\int_{\partial C_3} B = \int_{\partial C_3} B(\mathbf{r}, t) = \int_{\partial C_3} B(\mathbf{r}, t_0 + \Delta t) - \int_{\partial C_3} B(\mathbf{r}, t_0), \tag{91}$$

and

$$\int_{\partial C_3} E \wedge \mathrm{d}t = \int_{\partial S} \int_T E(\mathbf{r}, t)\, \mathrm{d}t. \tag{92}$$

Thus, Eq. (89) is reduced to

$$\int_{\partial C_3} B + \int_{\partial C_3} E \wedge \mathrm{d}t = \int_S [B(\mathbf{r}, t_0 + \Delta t) - B(\mathbf{r}, t_0)] + \int_{\partial S} \int_T E(\mathbf{r}, t) \wedge \mathrm{d}t = 0. \tag{93}$$

Let us assume that we have an infinitesimal time interval $\Delta t \to 0$. In this case Eq. (93) can be simplified as

$$\int_S [B(\mathbf{r}, t_0 + \Delta t) - B(\mathbf{r}, t_0)] = -\Delta t \int_{\partial S} E(\mathbf{r}, t_0),$$

which leads to the final equation:

$$\frac{\partial}{\partial t} \int_S B(\mathbf{r}, t)|_{t_0} = -\int_{\partial S} E(\mathbf{r}, t_0). \tag{94}$$

We can recall now that $B = \mathbf{B} \cdot \mathrm{d}\boldsymbol{\Sigma}$ is a magnetic induction flux 2-form, and $E = \mathbf{E} \cdot \mathbf{dr}$ is an electric voltage 1-form. Therefore, Eq. (94) represents a conventional Faraday's law for electromagnetic induction:

$$\int_{\partial S} \mathbf{E}(\mathbf{r}, t_0) \cdot \mathbf{dr} = -\frac{\partial}{\partial t} \int_S \mathbf{B}(\mathbf{r}, t) \cdot \mathrm{d}\boldsymbol{\Sigma}|_{t_0}. \tag{95}$$

7.2. Integral formulation of Ampere's law

In a similar way we can demonstrate that the differential form equation for Maxwell's field (70) results in Ampere's electromagnetic law. Indeed, let us reproduce Eq. (70) for convenience:

$$\mathrm{d} \wedge M = \gamma^e. \tag{96}$$

Integrating both sides of this equation over the geometrical element $C_3 = T \times S$ and taking into account the general Stokes's theorem (87), we obtain:

$$\int_{C_3} \mathrm{d} \wedge M = \int_{\partial C_3} M = \int_{\partial C_3} \gamma^e. \tag{97}$$

Substituting expressions (68) and (72) for Maxwell's field M and four-current γ^e in Eq. (97), we have:

$$\int_{\partial C_3} (D - H \wedge \mathrm{d}t) = \int_{C_3} [q\mathrm{d}v - j \wedge \mathrm{d}t], \tag{98}$$

where

$$j = \mathbf{j} \cdot \mathrm{d}\boldsymbol{\Sigma} \tag{99}$$

is an electric current 2-form.

Assuming that we have an infinitesimal time interval $\Delta t \rightarrow 0$, and proceeding in analogy with the force field equations above, we can calculate the integrals in Eq. (88) as

$$\int_{\partial C_3} (D - H \wedge \mathrm{d}t) = \int_S [D(\mathbf{r}, t_0 + \Delta t) - D(\mathbf{r}, t_0)] - \Delta t \int_{\partial S} H(\mathbf{r}, t_0) \tag{100}$$

and

$$\int_{T \times S} q\mathrm{d}v = 0, \quad \int_{T \times S} j \wedge \mathrm{d}t = -\Delta t \int_S j(\mathbf{r}, t_0). \tag{101}$$

Substituting expressions (100) and (101) back into Eq. (98), we arrive at the final integral form of the first Maxwell's equation:

$$\frac{\partial}{\partial t} \int_S D(\mathbf{r}, t)|_{t_0} - \int_{\partial C} H(\mathbf{r}, t_0) = -\int_S j(\mathbf{r}, t_0). \tag{102}$$

Taking into account that $D = \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma}$ is an electric displacement flux 2-form, $H = \mathbf{H} \cdot \mathrm{d}\mathbf{r}$ is magnetic work 1-form, and $j = \mathbf{j} \cdot \mathrm{d}\boldsymbol{\Sigma}$ is an electric current 2-form, we obtain the classical integral formulation of Ampere's law:

$$\int_{\partial S} \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \int_S (\mathbf{j} \cdot \mathrm{d}\boldsymbol{\Sigma}) + \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot \mathrm{d}\boldsymbol{\Sigma}. \tag{103}$$

7.3. Integral equations for Maxwell's field and force field in the frequency domain

We can obtain the integral equations for Maxwell's field and force field in the frequency domain by introducing frequency domain differential forms:

$$\{E(\mathbf{r},\omega), D(\mathbf{r},\omega), H(\mathbf{r},\omega), B(\mathbf{r},\omega), j(\mathbf{r},\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{E(\mathbf{r},\omega), D(\mathbf{r},\omega), H(\mathbf{r},\omega), B(\mathbf{r},\omega), j(\mathbf{r},\omega)\}e^{-i\omega t}dt. \quad (104)$$

Applying the Fourier transform (104) to Eqs (94) and (102), we obtain:

$$\int_{\partial S} E(\mathbf{r},\omega) = i\omega \int_S B(\mathbf{r},\omega) \quad (105)$$

and

$$\int_{\partial S} H(\mathbf{r},\omega) = \int_S j(\mathbf{r},\omega) - i\omega \int_S D(r,\omega). \quad (106)$$

Equations (105) and (106) provide integral representations of Maxwell's differential form equations in the frequency domain.

8. Numerical Modeling Using Differential Forms

There are several ways to obtain discrete analogs of Maxwell's equations. In the vast majority of numerical algorithms, the model region is discretized into a number of prisms as shown in Figure 1 (Zhdanov, 2002).

A Cartesian coordinate system is defined with the z axis directed downward, and the x axis directed to the right. The indices i, k, and l are used to number the grid points in the x, y, and z directions, respectively. The electromagnetic parameters, σ, μ, and ε are assumed to be constant within each elementary prism. We denote this grid by Σ:

$$\Sigma = \left\{ (x_i, y_k, z_l) \begin{array}{ll} x_1 = x',\ x_{N_I} = x'' & x_{i+1} = x_i + \Delta x_i \\ & i = 1,2,\ldots,N_I \\ y_1 = y',\ y_{N_K} = y'' & y_{k+1} = y_k + \Delta y_k \\ & k = 1,2,\ldots,N_K \\ z_1 = z',\ z_{N_L} = z'' & z_{l+1} = z_l + \Delta z_l \\ & l = 1,2,\ldots,N_L \end{array} \right\}.$$

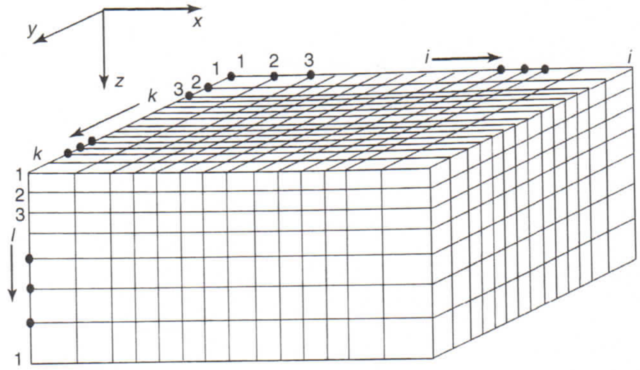

Figure 1 The model region is discretized into a number of prisms. The indices i, k, and l are used to number the grid points in the x, y, and z directions, respectively. The electromagnetic parameters, σ, μ, and ε are assumed to be constant within each elementary prism.

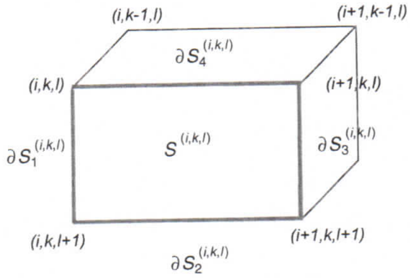

Figure 2 Discretization of the electromagnetic differential forms on a rectangular grid.

Let us consider one prismatic cell of the grid (Figure 2). We denote by S a face of the prism, bounded by a rectangular contour ∂S. We consider the integral Eq. (105) for EM differential forms written for a face of the prism.

We can evaluate the contour integral in Eq. (105) as a sum of four integrals over the edges of the face of the prism:

$$\int_{\partial S_1} E(\mathbf{r},\omega) + \int_{\partial S_2} E(\mathbf{r},\omega) + \int_{\partial S_3} E(\mathbf{r},\omega) + \int_{\partial S_4} E(\mathbf{r},\omega) = \mathrm{i}\omega \int_S B(\mathbf{r},\omega). \tag{107}$$

For example, at the prism face $S^{(i,k,l)}$ parallel to the vertical plane xz (see Figure 2) and having the upper left corner at the node (i,k,l) the integral

Eq. (107) can be written as follows:

$$\int_{\partial S_1^{(i,k,l)}} E(\mathbf{r},\omega) + \int_{\partial S_2^{(i,k,l)}} E(\mathbf{r},\omega) + \int_{\partial S_3^{(i,k,l)}} E(\mathbf{r},\omega) + \int_{\partial S_4^{(i,k,l)}} E(\mathbf{r},\omega) = \mathrm{i}\omega \int_{S^{(i,k,l)}} B(\mathbf{r},\omega). \tag{108}$$

We denote by $E_m(i,k,l)$ a voltage along the edge $\partial S_m^{(i,k,l)}$, $m = 1, 2, 3, 4$,

$$E_m(i,k,l) = \int_{\partial S_m^{(i,k,l)}} E(\mathbf{r},\omega),$$

and by $B(i,k,l)$ a magnetic induction flux through the prism's face $S^{(i,k,l)}$,

$$B(i,k,l) = \int_{S^{(i,k,l)}} B(\mathbf{r},\omega).$$

Using these notations, we can write Eq. (108) in the form:

$$\sum_{m=1}^{4} E_m(i,k,l) = B(i,k,l). \tag{109}$$

Similar algebraic expressions can be obtained for other faces of an elementary prism. We derive analogous formulae using Eq. (106) for EM differential forms written for a face of the prism. Combining all these algebraic equations together, we arrive at a full system of linear equations for the discretized values of the flux and voltage (work) of an electromagnetic field on a rectangular grid.

It is important to emphasize that the system of algebraic equations for the fluxes and voltages (work) of an electromagnetic field, derived above, provides an exact representation of the original system of Maxwell's equations for the differential forms. At the same time, any discretization of the classical system of Maxwell's equations for the vector fields based on finite difference or finite element methods results in some approximate representation of the vector fields. This property of the numerical methods based on differential form equations opens a possibility for developing a very accurate technique for electromagnetic modeling, especially in the case of a high conductivity contrast.

Another advantage of modeling based on the differential form equations is that the corresponding fluxes and voltages (work) of EM fields are continuous on the faces and edges, respectively, of homogeneous prisms.

9. Conclusions

We have demonstrated in this paper that the major differential equations for nonstationary vector fields can be expressed in a very simple form (47). The simplicity and symmetry of this equation indicates that the differential forms provide a natural representation for vector fields in four-dimensional space, E_4. Note that these forms are introduced as a linear combination of the flux of the vector field $\mathbf{D}$ through a vector element of the surface, $D = \mathbf{D} \cdot d\mathbf{\Sigma}$, and a time differential multiplied by the work of the vector field $\mathbf{H}$ along a vector element of a line, $H = \mathbf{H} \cdot \mathbf{dr}$. The differential equations for a pair of arbitrary vector fields in a four-dimensional Euclidean space have a structure identical to Maxwell's system of equations. Therefore, the basic laws of the classical theory of electromagnetic fields are encoded in the mathematical structure of the differential forms.

An important feature of the differential form of Maxwell's equations (79) and (80) is that they describe the relationships between the elementary fluxes and the work of the different EM field components, while the original Maxwell's equations (62)–(65) deal with the vectors of the EM fields themselves. Thus, the new mathematical form of Maxwell's equations emphasizes the importance of the fluxes and the work of the EM field. We should conclude that, the flux of the field through a given surface and the work of the field along a given path indeed represent the most important physical entities which are studied and measured in geophysical experiments. That is why the new form of Maxwell's equations (79) and (80) appears to be extremely well suited for the description of geophysical EM phenomena.

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