# Generalized joint inversion of multimodal geophysical data using Gramian constraints

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[1] We introduce a new approach to the joint inversion of multimodal geophysical data using Gramian spaces of model parameters and Gramian constraints, computed as determinants of the corresponding Gram matrices of the multimodal model parameters and/or their attributes. We demonstrate that this new approach is a generalized technique that can be applied to the simultaneous joint inversion of any number and combination of geophysical datasets. Our approach includes as special cases those extant methods based on correlations and/or structural constraints of the multimodal model parameters. As an illustration of this new approach, we present a model study relevant to exploration under cover for iron oxide copper-gold (IOCG) deposits, and demonstrate how joint inversion of gravity and magnetic data is able to recover alteration associated with IOCG mineralization. **Citation:** Zhdanov, M. S., A. Gribenko, and G. Wilson (2012), Generalized joint inversion of multimodal geophysical data using Gramian constraints, *Geophys. Res. Lett.*, *39*, L09301, doi:10.1029/2012GL051233.

## 1. Introduction

[2] Different geophysical fields provide information about different physical properties of the earth. Multiple geophysical surveys spanning gravity, magnetic, electromagnetic, and seismic methods are often interpreted to infer geology from models of different physical properties. In many cases, the various geophysical data are complimentary, making it natural to consider a formal mathematical framework for their joint inversion to a shared earth model. There are different approaches to joint inversion. The simplest case of joint inversion is where the physical properties are identical between different geophysical methods [*Jupp and Vozoff*, 1975].

[3] In other cases, joint inversion may infer theoretical, empirical or statistical correlations between different physical properties [*Chen et al.*, 2007]. In cases where the physical properties are not correlated but nevertheless have similar structural constraints, joint inversion can be formulated as a minimization of the cross-gradients between different physical properties [*Haber and Oldenburg*, 1997; *Gallardo and Meju*, 2003, 2007, 2011].

[4] Note that, in practical applications, the empirical or statistical correlations between different physical properties may exist, but their specific form may be unknown. In addition, there could be both analytical and structural correlations between different attributes of the model parameters. There remains a need to develop a method of joint inversion which would not require a priori knowledge about specific empirical or statistical relationships between the different model parameters and/or their attributes.

[5] In this letter, we address this problem by introducing a unified approach to joint inversion using Gramian constraints, which are based on the minimization of the determinant of the Gram matrix of a system of different model parameters (a Gramian). The basic underlying idea of this approach is that the Gramian provides a measure of correlation between the model parameters. By imposing an additional requirement of the minimum of the Gramian, we arrive at the solution of the joint multimodal inverse problem with the enhanced correlation between the different model parameters and/or their attributes. This unified approach is general, as it can be shown that extant methods based on correlations and/or structural constraints are special case reductions.

## 2. Gramian Spaces of Model Parameters

[6] In general, we can consider the modeling of multiple geophysical data as the operator relationships:

$$A^{(i)}\left(m^{(i)}\right) = d^{(i)},\ i = 1, 2, \ldots, n, \tag{1}$$

where, in a general case, $A^{(i)}$ is a nonlinear operator, $m^{(i)}$ ($i$ = 1, 2, 3, …, $n$) are the unknown model parameters which form a complex Hilbert space of model parameters, $M$, with an $L_2$ norm defined by the corresponding inner product:

$$\left(m^{(i)}, m^{(j)}\right)_M = \int_V m^{(i)}(\mathbf{r}) m^{(j)*}(\mathbf{r})\, dv,\ \ \left\|m^{(i)}\right\|_M^2 = \left(m^{(i)}, m^{(i)}\right)_M. \tag{2}$$

In equation (2), $\mathbf{r}$ is a radius vector defined within a volume, $V$; asterisk * denotes the complex conjugate. Note that $d^{(i)}$ are different data that belong to a complex Hilbert space of data, $D$, with an $L_2$ norm defined by the corresponding inner product:

$$\left(d^{(i)}, d^{(j)}\right)_D = \int_S d^{(i)}(\mathbf{r}) d^{(j)*}(\mathbf{r}) ds,\ \ \left\|d^{(i)}\right\|_D^2 = \left(d^{(i)}, d^{(i)}\right)_D, \tag{3}$$

where $S$ is an observation surface. Let us consider two arbitrary functions from the model space, $p(\mathbf{r})$, $q(\mathbf{r}) \in M$. We introduce a new inner product operation, $(p, q)_{G^{(n)}}$, between

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two functions, $p$ and $q$, as the determinant of the following matrix:

$$(p,q)_{G^{(n)}} = \begin{vmatrix} (m^{(1)},m^{(1)})_M & (m^{(1)},m^{(2)})_M & \dots & (m^{(1)},m^{(n-1)})_M & (m^{(1)},q)_M \\ (m^{(2)},m^{(1)})_M & (m^{(2)},m^{(2)})_M & \dots & (m^{(2)},m^{(n-1)})_M & (m^{(2)},q)_M \\ \dots & \dots & \dots & \dots & \dots \\ (m^{(n-1)},m^{(1)})_M & (m^{(n-1)},m^{(2)})_M & \dots & (m^{(n-1)},m^{(n-1)})_M & (m^{(n-1)},q)_M \\ (p,m^{(1)})_M & (p,m^{(2)})_M & \dots & (p,m^{(n-1)})_M & (p,q)_M \end{vmatrix}, \quad (4)$$

where all properties of the inner product hold:

$$(p,q)_{G^{(n)}} = (q,p)^*_{G^{(n)}}, \quad (5)$$

$$\left(\alpha_1 p^{(1)} + \alpha_2 p^{(2)}, q\right)_{G^{(n)}} = \alpha_1 \left(p^{(1)}, q\right)_{G^{(n)}} + \alpha_2 \left(p^{(2)}, q\right)_{G^{(n)}}, \quad (6)$$

$$(p,p)_{G^{(n)}} \geq 0. \quad (7)$$

The last property (7) follows from the fact that the norm square of a function, $||p||^2_{G^{(n)}}$, is equal to the determinant, $G(m^{(1)}, m^{(2)}, ...., m^{(n-1)}, p)$, of the Gram matrix of a set of functions, $(m^{(1)}, m^{(2)}, ...., m^{(n-1)}, p)$, which is called a *Gramian*:

$$\|p\|^2_{G^{(n)}} = (p,p)_{G^{(n)}} = G(m^{(1)},m^{(2)}, ...., m^{(n-1)},p) = \begin{vmatrix} (m^{(1)},m^{(1)})_M & (m^{(1)},m^{(2)})_M & \dots & (m^{(1)},m^{(n-1)})_M & (m^{(1)},p)_M \\ (m^{(2)},m^{(1)})_M & (m^{(2)},m^{(2)})_M & \dots & (m^{(2)},m^{(n-1)})_M & (m^{(2)},p)_M \\ \dots & \dots & \dots & \dots & \dots \\ (m^{(n-1)},m^{(1)})_M & (m^{(n-1)},m^{(2)})_M & \dots & (m^{(n-1)},m^{(n-1)})_M & (m^{(n-1)},p)_M \\ (p,m^{(1)})_M & (p,m^{(2)})_M & \dots & (p,m^{(n-1)})_M & (p,p)_M \end{vmatrix}. \quad (8)$$

The Gramian satisfies Gram's inequality:

$$G(m^{(1)},m^{(2)}, ...., m^{(n-1)},p) \geq 0. \quad (9)$$

Note that equality holds in (9) if the system of functions $(m^{(1)}, m^{(2)}, ...., m^{(n-1)}, p)$ is linearly dependent.

[7] We introduce a Gramian space of the model parameters, $G^{(n)}$, as the Hilbert space formed by the integrable functions, defined within a volume, $V$, with the inner product operation defined by equation (4). The main property of the Gramian space is that the norm of a function, $p$, in the Gramian space provides a measure of correlation between the function and the model parameters, $m^{(1)}, m^{(2)}, ...., m^{(n-1)}$.

[8] In a similar way, one can introduce a Gramian space $G^{(j)}$ where inner product is defined in a similar manner to equation (4), with the only difference that functions $p$ and $q$ are located within the row and column with number $j$, respectively:

$$(p,q)_{G^{(j)}} = \begin{vmatrix} (m^{(1)},m^{(1)})_M & (m^{(1)},m^{(2)})_M & \dots & (m^{(1)},q)_M & (m^{(1)},m^{(n)})_M \\ \dots & \dots & \dots & \dots & \dots \\ (p,m^{(1)})_M & (p,m^{(2)})_M & \dots & (p,q)_M & (p,m^{(n)}) \\ \dots & \dots & \dots & \dots & \dots \\ (m^{(n)},m^{(1)})_M & (m^{(n)},m^{(2)})_M & \dots & (m^{(n)},q)_M & (m^{(n)},m^{(n)})_M \end{vmatrix}. \quad (10)$$

In the Gramian space $G^{(j)}$, the norm square of a function, $||p||^2_{G^{(j)}}$, is equal to the Gramian of a set of functions, $(m^{(1)}, m^{(2)}, ...., m^{(j-1)}, p, m^{(j+1)}, ....m^{(n)})$:

$$\|p\|^2_{G^{(j)}} = (p,p)_{G^{(j)}} = G(m^{(1)},m^{(2)}, ....m^{(j-1)},p,m^{(j+1)},\dots,m^{(n)}). \quad (11)$$

Therefore, the norm of the function in the Gramian space, $G^{(j)}$, provides a measure of correlation between this function and all other model parameters, with the exception of parameter $m^{(j)}$. Note that this Gramian norm has the following property:

$$\|m^{(i)}\|^2_{G^{(i)}} = \|m^{(j)}\|^2_{G^{(j)}}, \quad (12)$$

for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$. The relationship stated in equation (12) follows directly from the definition of the Gramian norm, equation (11). Equation (12) demonstrates that all functions have the same norm in the corresponding Gramian spaces, $G^{(j)}$, $j = 1, 2, \dots, n$.

## 3. Gramian Spaces of Model Parameter Transforms

[9] The use of Gramian constraints can be generalized to make it possible to introduce any function of the model parameters. We do this by introducing a transform operator, $T$, of the model parameters from model space, $M$, to the transformed model space, $M_T$:

$$f = Tp, \quad (13)$$

$$g = Tq, \quad (14)$$

where $p, q \in M, f, g \in M_T$. The transform operator, $T$, can be chosen as a differential operator (e.g., gradient or Laplacian), an absolute value of the model parameters or their derivatives, a Fourier transform, a logarithm, an exponential, or any other transform which emphasizes specific properties of the models. We consider all transformations as attributes of the model parameters, because they are defined as some functions of the model parameters. Let us consider two arbitrary functions from the transformed model space with a given inner product operation:

$$(f,g)_{M_T} = \int_V f(\mathbf{r}) g^*(\mathbf{r}) dv. \quad (15)$$

We can introduce an inner product operation, $(f, g)_{G_T^{(n)}}$, between the two functions as the matrix determinant:

$$(f,g)_{G_T^{(n)}} = \begin{vmatrix} (Tm^{(1)},Tm^{(1)})_{M_T} & (Tm^{(1)},Tm^{(2)})_{M_T} & \dots & (Tm^{(1)},g)_{M_T} \\ (Tm^{(2)},Tm^{(1)})_{M_T} & (Tm^{(2)},Tm^{(2)})_{M_T} & \dots & (Tm^{(2)},g)_{M_T} \\ \dots & \dots & \dots & \dots \\ (Tm^{(n-1)},Tm^{(1)})_{M_T} & (Tm^{(n-1)},Tm^{(2)})_{M_T} & \dots & (Tm^{(n-1)},g)_{M_T} \\ (f,Tm^{(1)})_{M_T} & (f,Tm^{(2)})_{M_T} & \dots & (f,g)_{M_T} \end{vmatrix}. \quad (16)$$

[10] The norm square of a transformed function, $\|Tp\|^2_{G_T^{(n)}}$, is equal to the Gramian of a system of transforms, ($Tm^{(1)}$, $Tm^{(2)}$, ...., $Tm^{(n-1)}$, $Tp$):

$$\|Tp\|^2_{G_T^{(n)}} = G(Tm^{(1)}, Tm^{(2)}, ...., Tm^{(n-1)}, Tp). \quad (17)$$

Therefore, the norm of the transformed function $p$ in the Gramian space provides a measure of correlation between the transform of this function and similar transforms of the model parameters, $Tm^{(1)}$, $Tm^{(2)}$, ...., $Tm^{(n-1)}$. Minimization of the norm, $\|Tp\|_{G_T^{(n)}}$, will result in multi-attributed models with correlated transforms of the model parameters.

## 4. Gramian Spaces of Model Parameter Gradients

[11] As an example of one class of model parameter transforms, we can consider the gradients of the model parameters. While there may not be any correlations between different model parameters, there may be structural correlations of their distributions, which can be related in a Gramian space of model parameter gradients. This is equivalent to the now widely used approach of minimizing the cross-gradients between different model parameters [*Gallardo and Meju*, 2003]. For example, we can select the operator, $T$, as the gradient operator, $\nabla$. We can determine the inner product of two arbitrary gradient functions from the model space of gradients, $\nabla p(\mathbf{r})$, $\nabla q(\mathbf{r}) \in M_\nabla$, as:

$$(\nabla p, \nabla q)_{M_\nabla} = \int_V (\nabla p(\mathbf{r}) \cdot \nabla q^*(\mathbf{r}))dv. \quad (18)$$

According to equations (16) and (17), the norm square, $\|\nabla p\|^2_{G_\nabla^{(n)}}$, of a gradient of a function in the corresponding Gramian space, $G_\nabla^{(n)}$, is equal to the Gramian of the system of gradients, $\nabla m^{(1)}$, $\nabla m^{(2)}$, ...., $\nabla m^{(n-1)}$, $\nabla p$:

$$\|\nabla p\|^2_{G_\nabla^{(n)}} = G(\nabla m^{(1)}, \nabla m^{(2)}, \ldots, \nabla m^{(n-1)}, \nabla p). \quad (19)$$

Therefore, the norm of the gradient of a function, $p$, in the Gramian space provides a measure of correlation between the gradient of this function and the gradients of the model parameters, $\nabla m^{(1)}$, $\nabla m^{(2)}$, ..., $\nabla m^{(n-1)}$. Minimization of this norm, $\|\nabla p\|_{G_\nabla^{(n)}}$, will result in multi-attributed models with correlated gradients, similar to minimization of the cross-gradients of the model parameters.

## 5. Regularized Joint Inversion of Multimodal Data With the Gramian Stabilizers

[12] For regularized joint inversion, we minimize a parametric functional with the Gramian stabilizers:

$$P^\alpha(m^{(1)}, m^{(2)}, ....m^{(n)}) = \sum_{i=1}^{n} \left\|A^{(i)}(m^{(i)}) - d^{(i)}\right\|_D^2 + \alpha c_1 \sum_{i=1}^{n} S^{(i)} + \alpha c_2 S_{G_T} \rightarrow \min \quad (20)$$

where $A^{(i)}(m^{(i)})$ are the predicted data, $\alpha$ is the regularization parameter, $S^{(i)}$ are smoothing or focusing stabilizing functionals of the corresponding model parameters [*Zhdanov*, 2002], $S_{G_T}$ is the Gramian stabilizing functional for transformed model parameters:

$$S_{G_T} = \|Tm^{(n)}\|^2_{G_T^{(n)}} = G(Tm^{(1)}, Tm^{(2)}, ...., Tm^{(n-1)}). \quad (21)$$

It is implied that the transform operator, $T$, may be the identity operator, and $c_1$ and $c_2$ are the weighting coefficients determining the weights of the different stabilizers in the parametric functional. At the initial stage of the inversion, coefficients $c_1$ and $c_2$ can be selected as unities. After calculating both the stabilizing and Gramian stabilizing functionals and comparing their magnitudes, it could be determined if an additional scaling is necessary. The coefficients $c_1$ and $c_2$ can be adjusted to bias either stabilizer. Note that, according to the properties of the norm, $\|\ldots\|_{G_T^{(n)}}$, in the Gramian space, $G_T^{(n)}$, minimization of this norm results in enforcing the correlation between different transforms (attributes) of the model parameters.

[13] To minimize parametric functional (20), we can construct the regularized conjugate gradient (RCG) method [*Zhdanov*, 2002], which for the $k$th iteration can be summarized as:

$$\mathbf{r}_k = A(\mathbf{m}_k) - \mathbf{d}, \quad (22a)$$
$$\mathbf{l}_k^\alpha = \mathbf{l}^\alpha(\mathbf{m}_k), \quad (22b)$$
$$\beta_k^\alpha = \|\mathbf{l}_k^\alpha\|^2 / \|\mathbf{l}_{k-1}^\alpha\|^2, \quad (22c)$$
$$\tilde{\mathbf{l}}_k^\alpha = \mathbf{l}_k^\alpha + \beta_k^\alpha \mathbf{l}_{k-1}^\alpha, \quad (22d)$$
$$\tilde{s}_k^\alpha = (\tilde{\mathbf{l}}_k^\alpha, \mathbf{l}_k^\alpha) / \left\{\|F_{m_k}\tilde{\mathbf{l}}_k^\alpha\|^2 + \alpha\|W\tilde{\mathbf{l}}_k^\alpha\|^2\right\}, \quad (22e)$$
$$\mathbf{m}_{k+1} = \mathbf{m}_k - \tilde{s}_k^\alpha \tilde{\mathbf{l}}_k^\alpha, \quad (22f) \quad (22)$$

where $\mathbf{d} = \left(d^{(1)}, d^{(2)}, \ldots, d^{(n)}\right)$ is the vector of observed data, $\mathbf{m} = \left(m_k^{(1)}, m_k^{(2)}, \ldots, m_k^{(n)}\right)$ is the vector of model parameters, $A(\mathbf{m}_k)$ is the vector of predicted data, $F_{m_k}$ is the linear operator of the Fréchet derivative of $A(\mathbf{m}_k)$, $W$ is a model weighting matrix, and $\mathbf{l}_k^\alpha = \left(l_k^{\alpha(1)}, l_k^{\alpha(2)}, \ldots, l_k^{\alpha(n)}\right)$ is the vector of the direction of steepest ascent. Coefficients $\beta_k^\alpha$ and $\tilde{s}_k^\alpha$ are scalars used to determine the conjugate direction and step lengths, respectively.

[14] Following *Zhdanov* [2002], expressions for the direction of steepest ascent, $l_k^{\alpha(i)}$, can be found from the first variation of the parametric functional (20):

$$\delta P^\alpha = 2\sum_{i=1}^{n} \left(F_m^{(i)}\delta m^{(i)}, A^{(i)}(m^{(i)}) - d^{(i)}\right)_D + 2\alpha\left(c_1 \sum_{i=1}^{n} \delta S^{(i)} + c_2 \delta S_{G_T}\right) = \sum_{i=1}^{n}\left(\delta m^{(i)}, l_k^{\alpha(i)}\right). \quad (23)$$

where $F_m^{(i)}$ is the linear operator of the Fréchet derivative of $A^{(i)}$. We now find the first variation of the Gramian stabilizing functional:

$$\delta S_{G_T} = \sum_{i=1}^{n} \delta_{\mathbf{m}^{(i)}} \|Tm^{(n)}\|^2_{G_T^{(n)}} = \sum_{i=1}^{n} \delta_{\mathbf{m}^{(i)}} \|Tm^{(i)}\|^2_{G_T^{(i)}} = 2\sum_{i=1}^{n}\left(\delta m^{(i)}, l_{G_T}^{(i)}\right), \quad (24)$$

**Table 1.** Physical Properties of Minerals That Form the Ternary System of the Synthetic IOCG Deposits

| Rock Type | Density (g/cc) | Susceptibility (SI) |
|---|---|---|
| Magnetite | 5.00 | 5.0 |
| Hematite & sulfides | 5.00 | 0.0 |
| Host | 2.65 | 0.0 |

where we take into account property (12) of the Gramian norm, and the first variation of the norm, $||Tm^{(i)}||^2_{G_T^{(i)}}$, is calculated as:

$$\delta_{\mathbf{m}^{(i)}} \left\| Tm^{(i)} \right\|^2_{G_T^{(i)}} = 2\left(\delta m^{(i)}, \sum_{j=1}^{n} (-1)^{i+j} G_{ij}^{Tm} F_T^* Tm^{(j)}\right) = 2\left(\delta m^{(i)}, l_{G_T}^{(i)}\right). \quad (25)$$

In this last equation, $G_{ij}^{Tm}$ is the corresponding minor of the Gram matrix $G(Tm^{(1)}, Tm^{(2)}, ...., Tm^{(n)})$ formed by eliminating column $i$ and row $j$, $F_T^*$ is the adjoint derivative of the transform operator, $T$, and vectors $l_{G_T}^{(i)}$ are the directions of steepest ascent for the Gramian stabilizing functionals, formed by the Gramian of the transformed model parameter:

$$l_{G_T}^{(i)} = 2\sum_{j=1}^{n} (-1)^{i+j} G_{ij}^{Tm} F_T^* Tm^{(j)}. \quad (26)$$

Substituting equation (26) into equation (23), we find the directions of steepest ascent of the parametric functional $P^{\alpha}$:

$$l^{\alpha(i)} = F_T^{(i)\star}\left(A^{(i)}(m^{(i)}) - d^{(i)}\right) + \alpha\left(c_1 l^{(i)} + c_2 l_{G_T}^{(i)}\right), \quad (27)$$

where $l^{(i)}$ are the directions of steepest ascent of the smoothing or focusing stabilizing functionals, which are explicitly defined in *Zhdanov* [2009].

[15] As per *Zhdanov* [2002], adaptive regularization is implemented to decrease the regularization parameter as the iterative process (22) proceeds until it is either terminated when the misfit reaches a desired level, or a maximum number of predetermined iterations is reached, or the misfit fails to decrease by a predetermined threshold between iterations. The interested reader can find a detailed explanation of the theory of the regularized conjugate gradient (RCG) method in *Zhdanov* [2002, 2009].

## 6. Model Study

[16] Mineral exploration has been driven towards covered terrains with little or no basement outcrop where the strategy is to obtain maximum value from pre-competitive public data and high resolution proprietary surveys. For example, the surface geology of the Gawler Craton in South Australia is characterized by an almost complete absence of basement outcrop, with variable mesoproterozoic-cretaceous cover to nearly 1300 m depth. Yet, the province is host to the world-class Olympic Dam iron oxide copper-gold (IOCG) deposit, and remains highly prospective [e.g., *Bastrakov et al.*, 2007]. For mineral explorers, the challenge is in discriminating between major mineralized IOCG systems and sub-economic or minor systems concealed by deep cover where minimal geologic and geochemical data are available.

[17] *Hanneson* [2003] described a simple method by which density and susceptibility could be used to infer the distribution, intensity, and proportions of hematite and magnetite alteration associated with IOCG mineralization. As per *Williams et al.* [2004], we assume that the primary controls on the physical properties are the three end-members: magnetite, hematite/sulfides, and barren host rock (Table 1); and that significant alteration will have a stronger effect on the physical properties than differences in the host lithology. We also assume that the magnetite component will include all susceptible minerals as their magnetite equivalents. Similarly, the hematite and sulfide component includes contributions from other dense non- or weakly-magnetic minerals. The inversion of total magnetic intensity (TMI) data assumes that the anomalous TMI is solely due to induced magnetization, and that the susceptibility must always be positive.

[18] Surface gravity and airborne TMI data were simulated for a synthetic earth model that contained two IOCG mineralization systems; one which would be considered of economic interest as contains magnetite alteration, and the other which would be considered uneconomic as it contains no magnetite alteration (Figure 1). Gravimeters were located on the surface every 200 m on a regular grid spanning 5000 m in the Easting, and 3000 m in the Northing. Total magnetic intensity (TMI) magnetometers were located on the same horizontal grid but at an elevation of 30 m above the ground. These survey designs simulate ground gravity and airborne magnetic surveys (Figure 2). For the purpose of this study, no noise was added to the data. However, we conducted other modeling and inversion studies which have confirmed the robustness of this approach to noisy data. The above described joint inversion methodology was applied with focusing regularization [*Zhdanov*, 2002], and Gramian constraints on the density and susceptibility with no other a priori information or constraints enforced. Initially, coefficients $c_1$ and $c_2$ were selected as unities. After scaling each of the model parameters $m^{(n)}$ by their corresponding maximum absolute values, both the stabilizing and Gramian stabilizing functionals displayed comparable magnitudes, and no additional scaling was necessary. An adaptive regularization parameter scheme was used with initial regularization parameter as a ratio of the norm of the residual functional to the norm of the stabilizing functions, and relaxation ratio of 0.9. Our choice of Gramian constraints on the density and susceptibility rather than their transforms (e.g., cross-gradients) is entirely appropriate given the known relations between the physical properties of the ternary system [*Hanneson*, 2003]. Note that, the advantage of the Gramian approach to the joint inversion is that it does not require a priori knowledge about specific empirical or statistical relationships between the different model parameters. In fact, we recover these relationships based on the results of the joint inversion, which will be illustrated below.

[19] From joint inversion, we estimate the density ($\rho$) and susceptibility ($\chi$) of each cell in the earth model. However, in mineral exploration, we are particularly interested in targeting potential mineralization systems. In the IOCG model study we have considered, we want to recover the fraction volumes of magnetite, hematite, and host rock. As described by *Hanneson* [2003], we have the weakly nonlinear relations

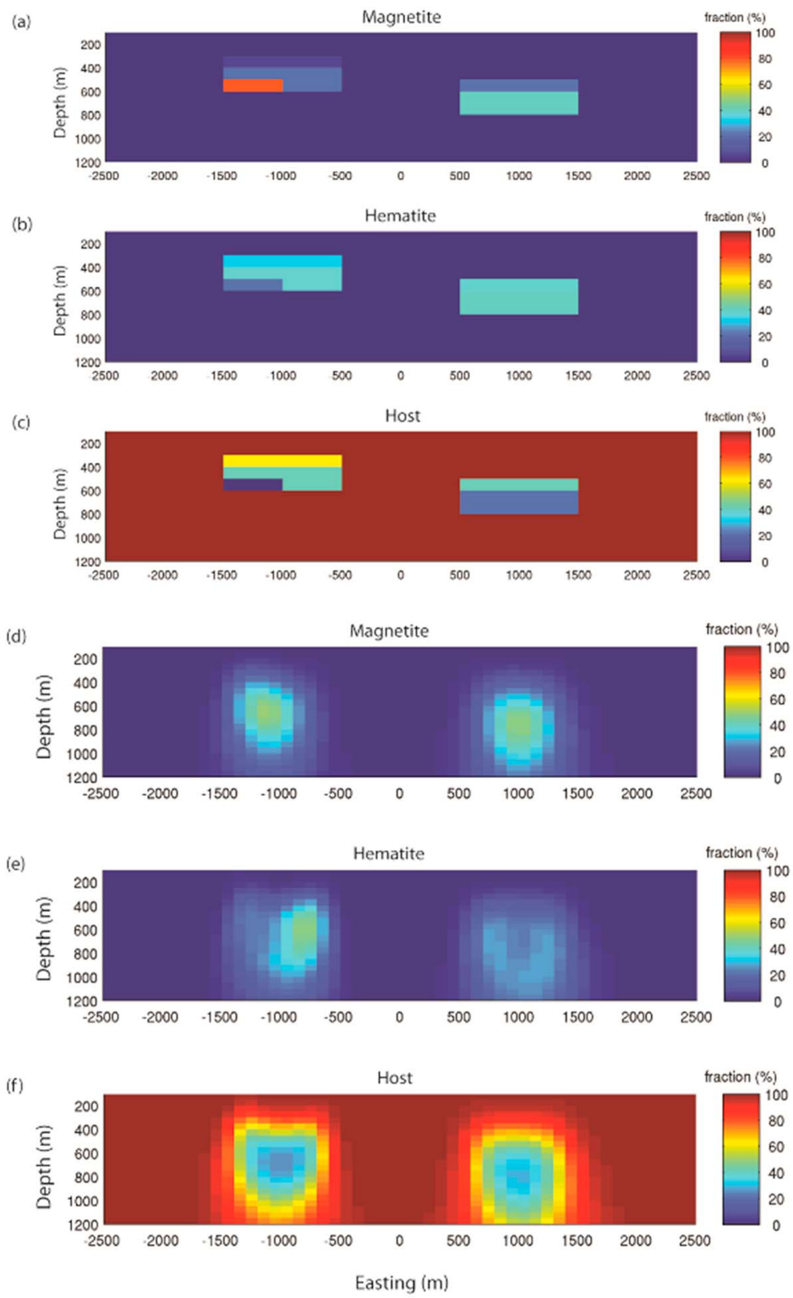


**Figure 1.** Vertical cross sections of the (a) magnetite, (b) hematite, and (c) host rock fraction volumes for the two synthetic IOCG mineralization systems for which gravity and magnetic data were simulated. Corresponding vertical cross sections of the (d) magnetite, (e) hematite, and (f) host rock fraction volumes of the IOCG mineralization systems as recovered from the joint inversion of synthetic gravity and magnetic data with focusing regularization and Gramian constraints on the density and susceptibility.

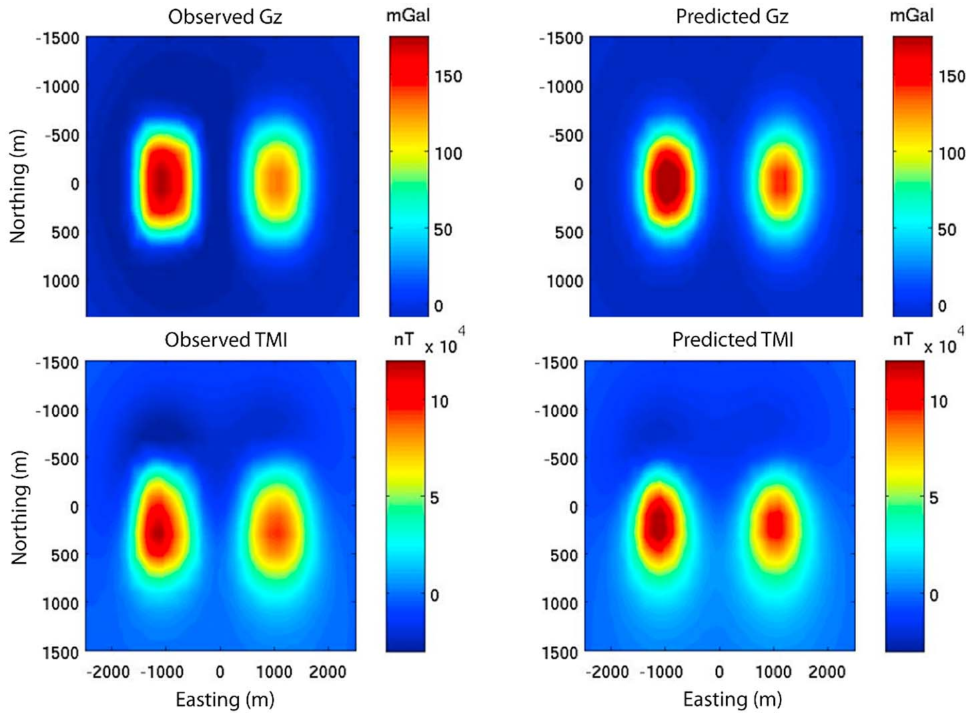


**Figure 2.** (left) Observed and (right) predicted (top) gravity and (bottom) total magnetic intensity data.

between the physical properties and fraction volumes (f) of the ternary system:

$$\chi = \chi^{\varphi}_{magnetite} f_{magnetite} + \chi_{hematite} f_{hematite} + \chi_{host} f_{host}, \quad (28)$$

$$\rho = \rho_{magnetite} f_{magnetite} + \rho_{hematite} f_{hematite} + \rho_{host} f_{host}, \quad (29)$$

$$1 = f_{magnetite} + f_{hematite} + f_{host}, \quad (30)$$

where $\varphi$ accommodates the nonlinear dependence of the susceptibility on the proportion of magnetite, and varies from 1.00 to 1.35. As per *Williams et al.* [2004], we can assume $\varphi = 1.00$ so equation (29) reduces to a linear system of three equations with three unknown fraction volumes which can be solved analytically. Applying this to our joint inversion results, we obtain 3D mineral models (Figure 1) for which we observe a magnetite and hematite/sulfide discrimination in the left target, and no such discrimination in the right target.

[20] Further, we can cross-plot the recovered densities and susceptibilities by a joint inversion for all cells in the earth model with the original physical properties (Figure 3a). We have also cross-plotted the densities and susceptibilities for all cells in the earth model as recovered from independent inversions of the gravity and magnetic data (Figure 3b). As expected, there is a continuum of the recovered physical properties. However, we note that, the physical properties recovered from joint inversion are characterized by clearly observed strong joint dependence (Figure 3a) with the trend, which follows the trend of the actual physical properties (red circles). At the same time, the physical properties recovered from independent inversions produce a "cloud" of points in the cross-correlation plot and practically have no trend with the actual physical properties (red circles). These plots clearly demonstrate a significant improvement in the correlation between the model parameters recovered by the joint inversion with the Gramian constraints in comparison with those obtained by the independent inversions.

[21] We note that additional minerals can also be added to equation (29) to make it an under-determined system [*Williams and Dipple*, 2007]. Further, equation (29) can be generalized to also include an effective medium model for the conductivity [e.g., *Zhdanov*, 2008] as would be recovered from the joint inversion of gravity, magnetic, and electromagnetic data. The resulting nonlinear system of equations for the fraction volumes is then solved via a nonlinear inversion.

## 7. Conclusions

[22] We have developed a generalized method for the joint inversion of multimodal geophysical data based on introduction of Gramian spaces of model parameters and Gramian constraints. The new method provides a unified approach to data fusion by considering the strength of the correlation between two or more sets of model parameters and/or their different attributes. This generalized method includes as special cases the existing methods based on structural constraints of the multimodal model parameters. Our model study relevant to IOCG exploration in the Gawler Craton of South Australia demonstrated how joint inversion of gravity and magnetic data could recover alteration associated with IOCG mineralization. Although we have demonstrated our method only for the joint inversion of gravity and magnetic data in this letter, we note that it can be extended to the simultaneous joint inversion of the results of any number or combination of other geophysical methods.

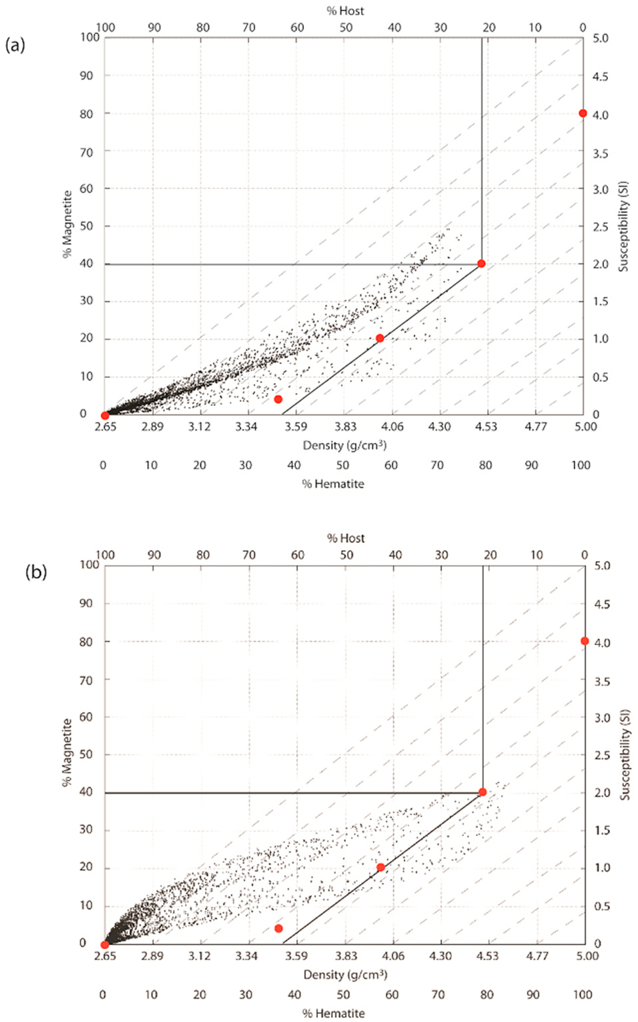


**Figure 3.** (a) Cross-plot of density and susceptibility, recovered from joint inversion with ternary mineralization of the synthetic IOCG systems superimposed. As expected given the ill-posedness of the joint inversion, there is a continuum of the recovered physical properties that trend with the actual physical properties (red circles). (b) Cross-plot of density and susceptibility recovered from independent inversions with ternary mineralization of the synthetic IOCG systems superimposed. Note that the continuum of the recovered physical properties do not trend with the actual physical properties (red circles).

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