Multinary Inversion for Tunnel Detection

Michael S. Zhdanov and Leif H. Cox

***Abstract*—We introduce multinary inversion to explicitly exploit the physical property contrasts between different objects and their host medium, e.g., between air-filled tunnels and their surrounding earth. Conceptually, multinary inversion is a generalization of binary inversion to multiple physical properties. However, unlike existing realizations of binary inversion which are solved using stochastic optimization methods, our realization of multinary inversion can be solved using deterministic optimization methods. This is significant as the method can be applied to both linear and nonlinear operators and easily extends to joint inversion of multimodal geophysical data. Using synthetic models of full-tensor gravity gradiometry data, multinary inversion is demonstrated to be robust for tunnel detection relative to the presence of significant geological noise.**

***Index Terms*—Binary inversion, inverse problems, multinary inversion, regularization, tunnel detection.**

I. Introduction

TRADITIONAL inverse methods characterize the model parameters of an examined medium by a function of the physical properties which varies continuously within known bounds. In order to obtain a unique and stable inverse solution, one can impose additional conditions on the model parameters, and these are usually enforced through minimum norm [1] or first derivative [2] ("smooth") stabilizing functionals. In many practical applications, the goal of inversion is to characterize targets with sharp boundaries and strong physical property contrasts between the targets and the host medium. For example, there exist significant physical property contrasts between an air-filled tunnel and the surrounding earth. Conventional smooth inversion of geophysical data cannot resolve these kinds of small discrete targets. An appropriate solution can be based on focusing regularization [3]–[5] which recovers models with sharp physical property boundaries and contrasts. However, the models produced from focusing inversion still contain a continuous distribution of the physical properties. In applications such as tunnel detection, the physical properties may be best described by a finite number of possible values. To this end, we introduce a novel method of inversion based on *multinary functions* and provide an example of its application to tunnel detection. By analogy with binary functions, we define multinary functions as those functions accepting a finite number of discrete values. Conceptually, *multinary inversion* is a generalization of binary inversion to multiple physical properties. However, unlike existing realizations of binary inversion which must be solved using stochastic optimization methods [6], [7], our realization of multinary inversion can be solved using deterministic optimization methods [3]. This is significant as the method can be applied to both linear and nonlinear operators and easily extends to the joint inversion of multimodal geophysical data [8].

Manuscript received August 25, 2012; revised October 8, 2012; accepted November 13, 2012. This work was supported in part by TechnoImaging and in part by The University of Utah's Consortium for Electromagnetic Modeling and Inversion.

M. S. Zhdanov is with the Department of Geology and Geophysics, The University of Utah, Salt Lake City, UT 84112 USA and also with TechnoImaging, Salt Lake City, UT 84107 USA (e-mail: mzhdanov@technoimaging.com).

L. H. Cox is with TechnoImaging, Salt Lake City, UT 84107 USA (e-mail: leif@technoimaging.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/LGRS.2012.2230433

II. Multinary Inversion

Inverse problems can be written as the discrete operator equation

$$\mathbf{d} = A(\mathbf{m}) \tag{1}$$

where $\mathbf{d} = [d_1, d_2, \ldots, d_{N_d}]$ is the N_d-length vector of observed data, $\mathbf{m} = [m_1, m_2, \ldots, m_{N_m}]$ is the N_m-length vector of model parameters, and A is a forward modeling operator, which, in a general case, may be linear or nonlinear. In this letter, as an example, we consider an application of our method to a linear gravity inverse problem. Inversion of (1) is ill posed and hence nonunique and unstable with respect to the noise in the data, and its solution requires regularization [1], [3]. We can solve inverse problem (1) using the Tikhonov parametric functional with a pseudoquadratic stabilizer [3]

$$P^{\alpha}(\mathbf{m}) = \phi(\mathbf{m}) + \alpha s(\mathbf{m}) \rightarrow \min \tag{2}$$

where $\phi(\mathbf{m})$ is a misfit functional

$$\phi(\mathbf{m}) = \|\mathbf{W}_d\mathbf{A}\mathbf{m} - \mathbf{W}_d\mathbf{d}\|_D^2 \tag{3}$$

with $\mathbf{W}_d$ being the data weighting matrix, and $s(\mathbf{m})$ is a stabilizing functional, which can be selected from the family of traditional smooth functionals [1], [2] or from the family of focusing functionals [3], [5]. The regularization parameter is introduced to provide balance (or bias) between the misfit and stabilizing functionals. Generally, the physical properties could accept any value within known (or physical) bounds. In this work, we introduce a model transform

$$\widetilde{m} = f(m) \tag{4}$$

such that the physical properties may be described by functions with a discrete number of values

$$\widetilde{m}_i = \left[\widetilde{m}_i^{(1)}, \widetilde{m}_i^{(2)}, \ldots, \widetilde{m}_i^{(P)}\right], \qquad i = 1, 2, \ldots, N_m \tag{5}$$

where we call $\widetilde{m}$ a *multinary function of order* P. The multinary function is chosen as monotonically increasing function such

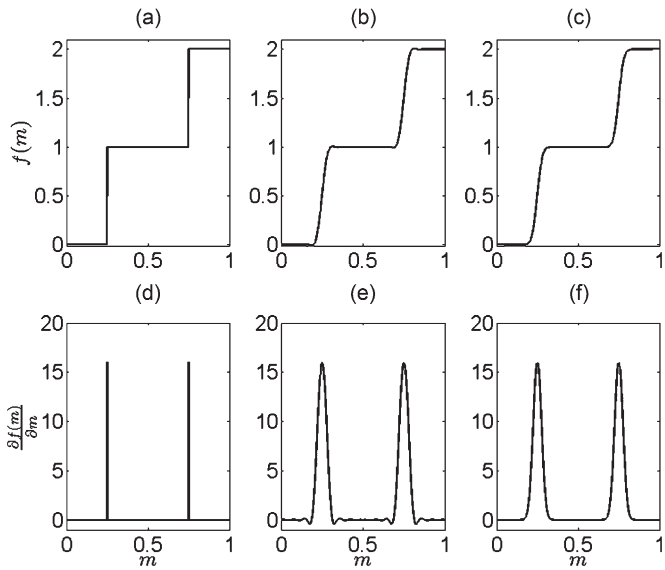

Fig. 1. Multinary functions f for the transform of continuous model parameters m for (a) Heaviside, (b) band-limited Heaviside, and (c) Gaussian staircase functions. Derivatives of the multinary functions are linear combinations of (d) delta, (e) band-limited delta, and (f) Gaussian functions.

that the derivative $\partial\widetilde{m}/\partial m$ is a known function so that the variation of (1) can be evaluated analytically

$$\tilde{\delta}d(\widetilde{m}) = \delta d(m) \times \left(\frac{\partial\widetilde{m}}{\partial m}\right)^{-1} \delta\widetilde{m} \tag{6}$$

thus enabling the parametric functional (2) to be minimized using any of the deterministic (gradient-based) optimization methods and related regularization techniques. In our work, we use the reweighted conjugate gradient method [3] to minimize the parametric functional (2). We should note that the original parametric functional (2) is always convex for linear forward operator A and has a unique global minimum [3]. The problem is then transformed into the multinary model space with a known monotonically increasing function, thus preserving the convexity of the optimization problem.

In the simplest case, the multinary function can be described by the steps of the staircase function $f(m)$

$$\widetilde{m}_i = f(m_i) = cm_i + \sum_{j=1}^{P} H\left(m_i - m_i^{(j)}\right) \tag{7}$$

where c is a small constant to avoid singularities in (6) and

$$H\left(m_i - m_i^{(j)}\right) = \begin{cases} 0, & m_i < m_i^{(j)} \\ 0.5, & m_i = m_i^{(j)} \\ 1, & m_i > m_i^{(j)} \end{cases} \tag{8}$$

is a Heaviside function [Fig. 1(a)]. For example, in Fig. 1, the steps of the staircase function have three values: zero, one, and two. The derivative of (7) is a linear combination of the delta functions [Fig. 1(d)]

$$\frac{\partial\widetilde{m}_i}{\partial m} = c + \sum_{j=1}^{P} \delta\left(m_i - m_i^{(j)}\right). \tag{9}$$

The practical difficulty with multinary function (7) is that the derivative (9) has singularities which prevent the use of gradient-based optimization methods. This has been the reason for binary inversion practitioners to use stochastic optimization methods for minimizing the parametric functional (e.g., [7]). However, these singularities can be avoided by introducing a band-limited Heaviside function [Fig. 1(b)]

$$\widetilde{m}_i = cm_i + \sum_{j=1}^{P} H_B\left(m_i - m_i^{(j)}\right) \tag{10}$$

$$H_B\left(m_i - m_i^{(j)}\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} T(\omega)\left[\pi\delta(\omega) - \frac{1}{i\omega}\right] e^{i\omega\left(m_i - m_i^{(j)}\right)} d\omega \tag{11}$$

where $T(\omega)$ is a symmetric and nonnegative function characterizing the band-limited filter in the frequency domain, such as a Tukey filter

$$T(\omega) = \begin{cases} \frac{1}{2}\left(1 + \cos\left(\frac{\pi\omega}{\beta}\right)\right), & |\omega| \le \beta \\ 0, & |\omega| \ge \beta. \end{cases} \tag{12}$$

We then obtain the continuous derivative of (10) as a superposition of band-limited delta functions [Fig. 1(b)]

$$\frac{\partial\widetilde{m}_i}{\partial m} = c + \sum_{j=1}^{P} \delta_B\left(m_i - m_i^{(j)}\right) \tag{13}$$

$$\delta_B\left(m_i - m_i^{(j)}\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} T(\omega) e^{-i\omega\left(m_i - m_i^{(j)}\right)} d\omega. \tag{14}$$

Alternatively, we can choose error functions [Fig. 1(c)]

$$\widetilde{m}_i = cm_i + \sum_{j=1}^{P} E\left(m_i - m_i^{(j)}\right) \tag{15}$$

$$E\left(m_i - m_i^{(j)}\right) = \operatorname{erf}\left(\frac{m_i - m_i^{(j)}}{\sqrt{2}\sigma_i}\right) \tag{16}$$

such that the derivative is a superposition of Gaussian functions [Fig. 1(f)]

$$\frac{\partial\widetilde{m}_i}{\partial m} = c + \sum_{j=1}^{P} G\left(m_i - m_i^{(j)}\right) \tag{17}$$

$$G\left(m_i - m_i^{(j)}\right) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{\frac{-\left(m_i - m_i^{(j)}\right)^2}{2\sigma_i^2}}. \tag{18}$$

Approximate representation (15) of multinary function (7) can be interpreted with a statistical analogy, where the Gaussian function (18) represents the probability density distribution of each discrete physical property m_i with the mean value $m_i^{(j)}$ and the standard deviation σ_i. Approximate representation (15) of multinary function (7) can then be interpreted as a cumulative density function of the physical properties.

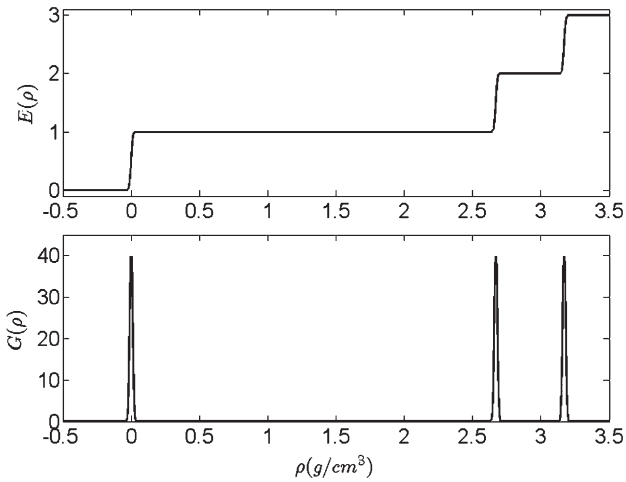

Fig. 2. Three-order multinary function used for tunnel detection described using an (upper panel) error function where the derivative is a (lower panel) Gaussian function. The three discrete densities used were 0.00 $\mathrm{g/cm^3}$ (air), 2.67 $\mathrm{g/cm^3}$ (earth), and 3.17 $\mathrm{g/cm^3}$ (mineralization).

III. Tunnel Detection

In this section, we will discuss the application of multinary inversion for tunnel detection. In recent years, every conceivable geophysical phenomenon, spanning the various seismic [9], [10], gravity [11], and electromagnetic [12]–[14] methods, has been investigated, evaluated, or employed to be applied for tunnel detection, including regularized inversion [3], parametric inversion [15], binary inversion [6], [7], contrast source inversion [16], [17], and level sets [18].

We can apply multinary inversion to explicitly exploit the physical property contrasts between air-filled tunnels and their surrounding earth. For example, the density of an air-filled tunnel is 0.00 $\mathrm{g/cm^3}$, and the density of the surrounding earth is approximately 2.67 $\mathrm{g/cm^3}$. To demonstrate the method for a ternary system, we also included a localized body with a density of 3.17 $\mathrm{g/cm^3}$. The multinary inversion was run with error function (16) and the minimum support (focusing) stabilizer. The multinary function was set to recover three discrete physical properties with densities of 0.00, 2.67, and 3.17 $\mathrm{g/cm^3}$ (Fig. 2). Note that the forward modeling operator for the gravity problem is a linear one, which ensures the convexity of the optimization problem described by (2). We considered a profile of the full-tensor gravity gradiometry (FTG) data [11] acquired 15 m above the earth's surface over a 2 m $\times$ 2 m tunnel buried 8 m below the surface [Fig. 3(d)]. As shown by the solid lines in Fig. 3(a)–(c), there are distinct responses due to the tunnel and the mineralization. To emulate geological noise, each cell in the earth model was randomly perturbed by ± 0.3 $\mathrm{g/cm^3}$ [Fig. 3(e)]. The FTG responses, including geological noise, are shown by the dashed lines in Fig. 3(a)–(c). The FTG responses, including geological noise, were then inverted using standard regularized inversion with a minimum norm stabilizer [Fig. 3(f)], standard regularized inversion with a focusing (minimum support) stabilizer [Fig. 3(g)], and multinary inversion with a focusing stabilizer [Fig. 3(h)].

As expected, the minimum norm inversion failed to detect the tunnel, instead recovering a smooth model that (significantly) underestimated the actual density distribution. The focusing

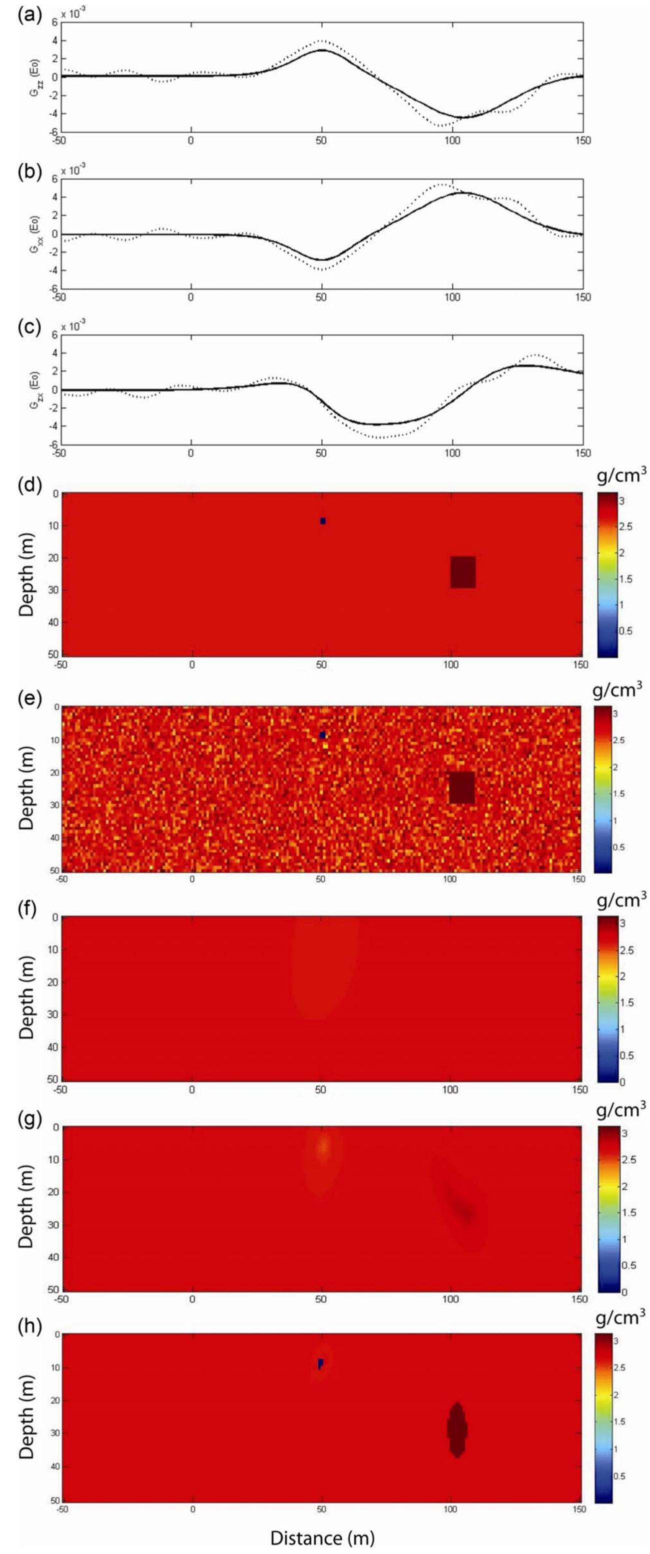

Fig. 3. FTG data acquired at 15-m elevation above the earth that is (solid line) noise free and (dashed line) contaminated with geological noise used for the inversion analysis for the (a) G_{zz}, (b) G_{xx}, and (c) G_{xz} components. (d) Earth model consists of a 2 m $\times$ 2 m air-filled tunnel (0 $\mathrm{g/cm^3}$) at 8-m depth and a 10 m $\times$ 10 m mineralized zone (3.17 $\mathrm{g/cm^3}$) in a uniform earth (2.67 $\mathrm{g/cm^3}$). (e) Geological noise was introduced by randomly perturbing the density of the earth by ± 0.3 $\mathrm{g/cm^3}$. (f) Regularized inversion result obtained from the use of the minimum norm stabilizer. (g) Regularized inversion result obtained from the use of the focusing (minimum support) stabilizer. (h) Multinary inversion result obtained from the use of an error function transform and the focusing (minimum support) stabilizer.

inversion successfully detected the air-filled tunnel and the high-density target but at a much lower contrast. The multinary inversion was able to image both the tunnel and the mineralization at their true densities. In a variety of other tests, we have established that multinary inversion is extremely robust to geological noise.

IV. Conclusion

We have introduced multinary inversion to solve the inverse problem for the models with the physical properties characterized by a finite number of discrete values. Multinary inversion can be applied to solve both linear and nonlinear inverse problems using deterministic optimization methods. We have used this method for solving the tunnel detection problem by explicitly exploiting the physical property contrasts between air-filled tunnels and their surrounding earth and have demonstrated the robustness of the developed method to geological noise. The method will easily extend to the joint inversion of multimodal geophysical data [8], and this is the subject of ongoing research.

References

[1] A. N. Tikhonov and V. Y. Arsenin, *Solutions of Ill-Posed Problems*. New York: V. H. Winston and Sons, 1977.

[2] S. C. Constable, R. L. Parker, and C. G. Constable, "Occam's inversion: A practical algorithm for generating smooth models from electromagnetic sounding data," *Geophysics*, vol. 52, no. 3, pp. 289–300, 1987.

[3] M. S. Zhdanov, *Geophysical Inverse Theory and Regularization Problems*. Amsterdam, The Netherlands: Elsevier, 2002.

[4] O. Portniaguine and M. S. Zhdanov, "Focusing geophysical inversion images," *Geophys.*, vol. 64, no. 3, pp. 874–887, Jun. 1999.

[5] M. S. Zhdanov, "New advances in regularized inversion of gravity and electromagnetic data," *Geophys. Prospect.*, vol. 57, no. 4, pp. 463–478, Jul. 2009.

[6] M. Bosch, A. Guillen, and P. Ledru, "Lithologic tomography: An application to geophysical data from the Cadomian belt of northern Brittany, France," *Tectonophysics*, vol. 331, no. 1/2, pp. 197–227, Feb. 2001.

[7] R. A. Krahenbuhl and Y. Li, "Inversion of gravity data using a binary formulation," *Geophys. J. Int.*, vol. 167, no. 2, pp. 543–556, Nov. 2006.

[8] M. S. Zhdanov, A. V. Gribenko, and G. A. Wilson, "3D joint inversion of multi-modal geophysical data using Gramian constraints," *Geophys. Res. Lett.*, vol. 39, no. 9, p. L09301, May 2012.

[9] T. Counts, G. Larson, A. C. Gurbuz, J. H. McClellan, and W. R. Scott, "Investigation of the detection of shallow tunnels using electromagnetic and seismic waves," in *Proc. SPIE*, 2007, vol. 6553, p. 655 31G.

[10] R. Miller, C. B. Park, J. Xia, J. Ivanov, D. W. Steeples, N. Ryden, R. F. Ballard, J. L. Llopis, T. S. Anderson, M. L. Moran, and S. A. Ketcham, "Tunnel detection using seismic methods," presented at the AGU Fall Meeting, Baltimore, MD, 1997, NS21A-07.

[11] A. J. Romaides, J. C. Battis, R. W. Sands, A. Zorn, D. O. Benson, and D. J. DiFrancesco, "A comparison of gravimetric techniques for measuring subsurface void signals," *J. Phys. D, Appl. Phys.*, vol. 34, no. 3, pp. 433–443, Feb. 2001.

[12] L. Lo Monte, D. Erricolo, F. Soldovieri, and M. C. Wicks, "Radio frequency tomography for tunnel detection," *IEEE Trans. Geosci. Remote Sens.*, vol. 48, no. 3, pp. 1128–1137, Mar. 2010.

[13] S. P. McKenna, K. B. Parkman, L. J. Perren, and J. R. McKenna, "Response of an electromagnetic gradiometer to a subsurface wire," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 12, pp. 4944–4953, Dec. 2011.

[14] J. A. Martinez-Lorenzo, C. M. Rappaport, and F. Quivira, "Physical limitations on detecting tunnels using underground-focusing spotlight synthetic aperture radar," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 1, pp. 65–70, Jan. 2011.

[15] R. L. Mackie, "Imaging of underground structures using HAARP," HAARP, Gakona, AK, Final Rep. AFRL-VS-TR-1999-1511, 1999.

[16] M. S. Zhdanov and V. V. Cherniak, "Automated method of solving the 2D inverse electromagnetic problem (in Russian)," *Izv. Akad. Nauk SSSR Fiz. Zemli*, vol. 296, no. 1, pp. 59–63, 1987.

[17] A. Abubakar and P. M. van den Berg, "The contrast source inversion method for location and shape reconstructions," *Inv. Prob.*, vol. 18, no. 2, pp. 495–510, Apr. 2002.

[18] O. Dorn and D. Lesselier, "Level set methods for inverse scattering," *Inv. Prob.*, vol. 22, no. 4, pp. R67–R131, Aug. 2006.