Multinary Inversion for Tunnel Detection

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Abstract—We introduce multinary inversion to explicitly exploit the physical property contrasts between different objects and their host medium, e.g., between air-filled tunnels and their surrounding earth. Conceptually, multinary inversion is a generalization of binary inversion to multiple physical properties. However, unlike existing realizations of binary inversion which are solved using stochastic optimization methods, our realization of multinary inversion can be solved using deterministic optimization methods. This is significant as the method can be applied to both linear and nonlinear operators and easily extends to joint inversion of multimodal geophysical data. Using synthetic models of full-tensor gravity gradiometry data, multinary inversion is demonstrated to be robust for tunnel detection relative to the presence of significant geological noise.

Index Terms—Binary inversion, inverse problems, multinary inversion, regularization, tunnel detection.

I. INTRODUCTION

TRADITIONAL inverse methods characterize the model parameters of an examined medium by a function of the physical properties which varies continuously within known bounds. In order to obtain a unique and stable inverse solution, one can impose additional conditions on the model parameters, and these are usually enforced through minimum norm [1] or first derivative [2] (“smooth”) stabilizing functionals. In many practical applications, the goal of inversion is to characterize targets with sharp boundaries and strong physical property contrasts between the targets and the host medium. For example, there exist significant physical property contrasts between an air-filled tunnel and the surrounding earth. Conventional smooth inversion of geophysical data cannot resolve these kinds of small discrete targets. An appropriate solution can be based on focusing regularization [3]–[5] which recovers models with sharp physical property boundaries and contrasts. However, the models produced from focusing inversion still contain a continuous distribution of the physical properties. In applications such as tunnel detection, the physical properties may be best described by a finite number of possible values. To this end, we introduce a novel method of inversion based on multinary functions and provide an example of its application to tunnel detection. By analogy with binary functions, we define multinary functions as those functions accepting a finite number of discrete values. Conceptually, multinary inversion is a generalization of binary inversion to multiple physical properties. However, unlike existing realizations of binary inversion which must be solved using stochastic optimization methods [6], [7], our realization of multinary inversion can be solved using deterministic optimization methods [3]. This is significant as the method can be applied to both linear and nonlinear operators and easily extends to the joint inversion of multimodal geophysical data [8].

II. MULTINARY INVERSION

Inverse problems can be written as the discrete operator equation

\[ \mathbf{d} = A(\mathbf{m}) \]  

where \( \mathbf{d} = [d_1, d_2, \ldots, d_{N_d}] \) is the \( N_d \)-length vector of observed data, \( \mathbf{m} = [m_1, m_2, \ldots, m_{N_m}] \) is the \( N_m \)-length vector of model parameters, and \( A \) is a forward modeling operator, which, in a general case, may be linear or nonlinear. In this letter, as an example, we consider an application of our method to a linear gravity inverse problem. Inversion of (1) is ill posed and hence nonunique and unstable with respect to the noise in the data, and its solution requires regularization [1], [3]. We can solve inverse problem (1) using the Tikhonov parametric functional with a pseudoquadratic stabilizer [3]

\[ P^\alpha(\mathbf{m}) = \phi(\mathbf{m}) + \alpha s(\mathbf{m}) \to \min \]  

where \( \phi(\mathbf{m}) \) is a misfit functional

\[ \phi(\mathbf{m}) = \| \mathbf{W}_d \mathbf{A} \mathbf{m} - \mathbf{d} \|_D^2 \]  

with \( \mathbf{W}_d \) being the data weighting matrix, and \( s(\mathbf{m}) \) is a stabilizing functional, which can be selected from the family of traditional smooth functionals [1], [2] or from the family of focusing functionals [3], [5]. The regularization parameter is introduced to provide balance (or bias) between the misfit and stabilizing functionals. Generally, the physical properties could accept any value within known (or physical) bounds. In this work, we introduce a model transform

\[ \tilde{\mathbf{m}} = f(\mathbf{m}) \]  

such that the physical properties may be described by functions with a discrete number of values

\[ \tilde{m}_i = [\tilde{m}_i^{(1)}, \tilde{m}_i^{(2)}, \ldots, \tilde{m}_i^{(P)}], \quad i = 1, 2, \ldots, N_m \]  

where we call \( \tilde{\mathbf{m}} \) a multinary function of order \( P \). The multinary function is chosen as monotonically increasing function such

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that the derivative $\partial \bar{m}/\partial m$ is a known function so that the variation of (1) can be evaluated analytically

$$\partial \bar{m} = \delta d(m) \times \left( \frac{\partial \bar{m}}{\partial m} \right)^{-1} \delta \bar{m}$$  \hspace{1cm} (6)

thus enabling the parametric functional (2) to be minimized using any of the deterministic (gradient-based) optimization methods and related regularization techniques. In our work, we use the reweighted conjugate gradient method [3] to minimize the parametric functional (2). We should note that the original parametric functional (2) is always convex for linear forward operator $A$ and has a unique global minimum [3]. The problem is then transformed into the multinary model space with a known monotonically increasing function, thus preserving the convexity of the optimization problem.

In the simplest case, the multinary function can be described by the steps of the staircase function $f(m)$

$$\bar{m}_i = f(m_i) = cm_i + \sum_{j=1}^{P} H \left( m_i - m_i^{(j)} \right)$$  \hspace{1cm} (7)

where $c$ is a small constant to avoid singularities in (6) and

$$H \left( m_i - m_i^{(j)} \right) = \begin{cases} 0, & m_i < m_i^{(j)} \\ 0.5, & m_i = m_i^{(j)} \\ 1, & m_i > m_i^{(j)} \end{cases}$$  \hspace{1cm} (8)

is a Heaviside function [Fig. 1(a)]. For example, in Fig. 1, the steps of the staircase function have three values: zero, one, and two. The derivative of (7) is a linear combination of the delta functions [Fig. 1(d)]

$$\frac{\partial \bar{m}_i}{\partial m} = c + \sum_{j=1}^{P} \delta \left( m_i - m_i^{(j)} \right) .$$  \hspace{1cm} (9)

The practical difficulty with multinary function (7) is that the derivative (9) has singularities which prevent the use of gradient-based optimization methods. This has been the reason for binary inversion practitioners to use stochastic optimization methods for minimizing the parametric functional (e.g., [7]). However, these singularities can be avoided by introducing a band-limited Heaviside function [Fig. 1(b)]

$$\bar{m}_i = cm_i + \sum_{j=1}^{P} H_B \left( m_i - m_i^{(j)} \right)$$  \hspace{1cm} (10)

$$H_B \left( m_i - m_i^{(j)} \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} T(\omega) \left[ \frac{\pi\delta(\omega) - \frac{1}{i\omega}}{e^{i\omega (m_i - m_i^{(j)})}} \right] d\omega$$  \hspace{1cm} (11)

where $T(\omega)$ is a symmetric and nonnegative function characterizing the band-limited filter in the frequency domain, such as a Tukey filter

$$T(\omega) = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{\omega}{\beta} \right) \right), & |\omega| \leq \beta \\ 0, & |\omega| \geq \beta. \end{cases}$$  \hspace{1cm} (12)

We then obtain the continuous derivative of (10) as a superposition of band-limited delta functions [Fig. 1(b)]

$$\frac{\partial \bar{m}_i}{\partial m} = c + \sum_{j=1}^{P} \delta_B \left( m_i - m_i^{(j)} \right)$$  \hspace{1cm} (13)

$$\delta_B \left( m_i - m_i^{(j)} \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} T(\omega)e^{-i\omega \left( m_i - m_i^{(j)} \right)} d\omega.$$  \hspace{1cm} (14)

Alternatively, we can choose error functions [Fig. 1(c)]

$$\bar{m}_i = cm_i + \sum_{j=1}^{P} E \left( m_i - m_i^{(j)} \right)$$  \hspace{1cm} (15)

$$E \left( m_i - m_i^{(j)} \right) = \text{erf} \left( \frac{m_i - m_i^{(j)}}{\sqrt{2}\sigma_i} \right)$$  \hspace{1cm} (16)

such that the derivative is a superposition of Gaussian functions [Fig. 1(f)]

$$\frac{\partial \bar{m}_i}{\partial m} = c + \sum_{j=1}^{P} G \left( m_i - m_i^{(j)} \right)$$  \hspace{1cm} (17)

$$G \left( m_i - m_i^{(j)} \right) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(m_i - m_i^{(j)})^2}{2\sigma_i^2}}.$$  \hspace{1cm} (18)

Approximate representation (15) of multinary function (7) can be interpreted with a statistical analogy, where the Gaussian function (18) represents the probability density distribution of each discrete physical property $m_i$ with the mean value $m_i^{(j)}$ and the standard deviation $\sigma_i$. Approximate representation (15) of multinary function (7) can then be interpreted as a cumulative density function of the physical properties.
III. TUNNEL DETECTION

In this section, we will discuss the application of multinary inversion for tunnel detection. In recent years, every conceivable geophysical phenomenon, spanning the various seismic [9], [10], gravity [11], and electromagnetic [12]–[14] methods, has been investigated, evaluated, or employed to be applied for tunnel detection, including regularized inversion [3], parametric inversion [15], binary inversion [6], [7], contrast source inversion [16], [17], and level sets [18].

We can apply multinary inversion to explicitly exploit the physical property contrasts between air-filled tunnels and their surrounding earth. For example, the density of an air-filled tunnel is 0.00 g/cm$^3$, and the density of the surrounding earth is approximately 2.67 g/cm$^3$. To demonstrate the method for a ternary system, we also included a localized body with a density of 3.17 g/cm$^3$. The multinary inversion was run with error function (16) and the minimum support (focusing) stabilizer. The multinary function was set to recover three discrete physical properties with densities of 0.00, 2.67, and 3.17 g/cm$^3$ (Fig. 2). Note that the forward modeling operator for the gravity problem is a linear one, which ensures the convexity of the optimization problem described by (2). We considered a profile of the full-tensor gravity gradiometry (FTG) data [11] acquired 15 m above the earth’s surface over a 2 m × 2 m tunnel buried 8 m below the surface [Fig. 3(d)]. As shown by the solid lines in Fig. 3(a)–(c), there are distinct responses due to the tunnel and the mineralization. To emulate geological noise, each cell in the earth model was randomly perturbed by ±0.3 g/cm$^3$ [Fig. 3(e)]. The FTG responses, including geological noise, are shown by the dashed lines in Fig. 3(a)–(c). The FTG responses, including geological noise, were then inverted using standard regularized inversion with a minimum norm stabilizer [Fig. 3(f)], standard regularized inversion with a focusing (minimum support) stabilizer [Fig. 3(g)], and multinary inversion with a focusing stabilizer [Fig. 3(h)].

As expected, the minimum norm inversion failed to detect the tunnel, instead recovering a smooth model that (significantly) underestimated the actual density distribution. The focusing

![Image](image_url)
inversion successfully detected the air-filled tunnel and the high-density target but at a much lower contrast. The multinary inversion was able to image both the tunnel and the mineralization at their true densities. In a variety of other tests, we have established that multinary inversion is extremely robust to geological noise.

IV. CONCLUSION

We have introduced multinary inversion to solve the inverse problem for the models with the physical properties characterized by a finite number of discrete values. Multinary inversion can be applied to solve both linear and nonlinear inverse problems using deterministic optimization methods. We have used this method for solving the tunnel detection problem by explicitly exploiting the physical property contrasts between air-filled tunnels and their surrounding earth and have demonstrated the robustness of the developed method to geological noise. The method will easily extend to the joint inversion of multimodal geophysical data [8], and this is the subject of ongoing research.

REFERENCES