# Inversion of gravity and gravity gradiometry data for density contrast surfaces using Cauchy-type integrals

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## Summary

We introduce a new method of modeling and inversion of potential field data generated by a density contrast surface. Our method is based on 3D Cauchy-type integral representation of the potential fields. Traditionally, potential fields are calculated using volume integrals of the domains occupied by anomalous masses subdivided into prismatic cells. This discretization is computationally expensive, especially in a 3D case. The Cauchy-type integral technique makes it possible to represent the gravity field and its gradients as surface integrals. This is especially significant in the solution of problems of modeling and inversion of gravity data for determining the depth to the basement. We demonstate our inversion method based on the Cauchy-type integral for several synthetic models. The results show that the new method is capable of providing high-resolution depth estimation for the sediment-to-basement interface.


## Introduction

There is a strong interest in developing effective methods of inverting gravity data for depth-to-basement and gravity contrast estimation. Large numbers of research papers have been published over the last decade on this subject (e.g., Barbosa et al., 1997, 1999a, b; Martins et al., 2011; Silva et al., 2010a, b). The conventional approach to solving depth-to-basement gravity inverse problems is based on parameterization of the earth's subsurface, containing the sedimentary pack, into prismatic cells with known horizontal dimensions and known density contrast, and on estimation of the cell's thicknesses.

In this paper, we present a novel approach to the solution of this problem based on 3D analogs of Cauchy-type integrals, introduced by Zhdanov (1980, 1984, 1988). These integrals extend to the 3D case all the major properties of classical Cauchy integrals of the theory of functions of complex variables. In our study, we apply the method of 3D Cauchy type integrals to solving both forward and inverse problems for a density contrast model. This type of models is used, for example, in the inversion of the gravity data for the depth-to-basement.

We have developed an inversion scheme to determine the density contrast surface from the observed potential field data if the densities of the sediments and basement are given. The gravity field and/or full tensor gravity gradiometry data can be used for the inversion.

## 3D analog of the Cauchy-type integral and its properties

The 3D analog of the Cauchy-type integral and its derivation was presented in Zhdanov (1988) as follows:

$$\mathbf{C}^s(\mathbf{r}',\boldsymbol{\varphi}) = -\frac{1}{4\pi}\iint_S \left[(\boldsymbol{n}\cdot\boldsymbol{\varphi})\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} + (\boldsymbol{n}\times\boldsymbol{\varphi})\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right]ds, \quad (1)$$

where $S$ is some closed surface bounding a domain $D$, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{r})$ is some vector function defined on the closed surface $S$, and $\mathbf{n}$ is the normal vector to the surface $S$ pointing outside $D$. The vector function $\boldsymbol{\varphi}$ is called the vector density of the Cauchy-type integral.

According to Zhdanov (1988), Cauchy-type integral formulas can be represented using matrix notations. In a Cartesian coordinate system $\{\mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_z\}$, we can represent the vectors $\mathbf{C}^s$, $\boldsymbol{\varphi}$, $\mathbf{n}$ and $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ as follows:

$$\mathbf{C}^s = \mathrm{C}^s_\alpha \mathbf{d}_\alpha, \boldsymbol{\varphi} = \varphi_\beta \mathbf{d}_\beta, \mathbf{n} = n_\gamma \mathbf{d}_\gamma, \quad (2)$$

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{r_\eta - r'_\eta}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{d}_\eta, \quad (3)$$

where $r_\eta = \eta$ ; $\alpha, \beta, \gamma = x, y, z$ , and we also use the convention that the twice recurring index indicates a summation over the index. Using these notations, we can write the scalar components of Cauchy-type integral as follows:

$$\mathrm{C}^s_\alpha = -\frac{1}{4\pi}\iint_S \Delta_{\alpha\beta\gamma\eta}\varphi_\beta\frac{r_\eta - r'_\eta}{|\mathbf{r}-\mathbf{r}'|^3}n_\gamma\, ds, \alpha, \beta, \gamma, \eta = x, y, z; \quad (4)$$

where the four-index $\Delta$-symbol is expressed in terms of the symmetric Kronecker symbol $\delta_{\alpha\beta}$ as:

$$\Delta_{\alpha\beta\gamma\eta} = \delta_{\alpha\beta}\delta_{\gamma\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\gamma\eta}; \delta_{\beta\eta} = \begin{cases}1, \alpha = \beta \\ 0, \alpha \neq \beta\end{cases}$$

## Cauchy-type representation of the gravity field and its gradient for a density contrast surface

Let us consider a model of the sediment-basement interface with a density contrast at some surface $\Gamma$ shown in Figure 1. We assume that surface $\Gamma$ is described by equation



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$z = h(x, y) - H_0$, and a horizontal plane $P$ is given by equation $z = -H_0$, where $H_0 \geq -h(x, y) \geq 0$, and:

$$h(x, y) - H_0 \to 0 \text{ for } \sqrt{x^2 + y^2} \to \infty,$$

where $H_0$ is a constant. Let us draw a sphere $O_R$ of radius $R$ with the center in the origin of the Cartesian system of coordinates. We denote by $\Gamma_R$ and $P_R$ the parts of the surfaces Γ and P, respectively, located within the sphere $O_R$. The gravity anomaly is caused by the density volume $D_R$: which is bounded by a closed surface, formed by $\Gamma_R$ and $P_R$ and the parts of the sphere $O_R$ between these two surfaces as shown in Figure 1.

It is demonstrated in Zhdanov (1988) that the gravity field caused by volume $D_R$ is expressed by:

$$\mathbf{g}(\mathbf{r}') = 4\pi\gamma_g\rho_0 \mathbf{C}^{\Gamma_R}(\mathbf{r}', (z + H_0)\mathbf{d}_z). \quad (5)$$

As a result, in the case where $\Gamma_{R\to\infty} \to P$ at infinity, the Cauchy-type integral in equation 5 is calculated along an infinitely extended surface Γ only.

In matrix notations, the gravity field caused by the density anomaly for a model shown in Figure 1 can be expressed as follows:

$$g_\alpha(\mathbf{r}') = -\gamma_g\rho_0 \iint_S \Delta_{\alpha z\gamma\eta} \frac{h(x, y)(r_\eta - r'_\eta)}{|\mathbf{r} - \mathbf{r}'|^3} b_\gamma \, dxdy, \alpha, \gamma, \eta = x, y, z; \quad (6)$$

Similarly, the gravity gradient can be written as:

$$g_{\alpha v}(\mathbf{r}') = \gamma_g\rho_0 \iint_S \Delta_{\alpha z\gamma\eta} \frac{h(x,y)}{|\mathbf{r}-\mathbf{r}'|^5} [3(r_v - r'_v)(r_\eta - r'_\eta) - |\mathbf{r} - \mathbf{r}'|\delta_{v\eta}] b_\gamma dxdy, \alpha, \gamma, \eta = x, y, z; \quad (7)$$

Equations 6 and 7 represent the gravity and gravity gradient fields in the form of Cauchy-type integrals over the density contrast surface corresponding to the sediment-basement interface. These expressions provide an analytical basis for a fast method of numerical modeling of gravity and gravity gradiometry data. Both of these two equations need to be discretized to be solved numerically. In the paper by Zhdanov and Liu (2013), the rectangular and triangular discretizations of the density contrast surface were introduced. Numerically, rectangular is simpler than triangular discretization. However, triangular discretization is demonstrated to have higher accuracy than rectangular. In this paper, we only use the scheme for rectangular discretization. For those who are interested in triangular discretization, please refer to Zhdanov and Liu (2013).

We can discretize the Cauchy-type integral for computing the gravity field and its gradient in equations (6) and (7) by dividing the horizontal plane *XY* into a grid of $N_m$ cells with constant discretization $\Delta_x$ and $\Delta_y$ in the $x$ and $y$ directions. As a result, within each cell $P_k$ ($k=1, 2, \ldots N_m$), the corresponding density contrast surface can be represented by a flat plane described by the following linear equation:

$$z = h(x, y) - H_0 = h^{(k)} - b_x^{(k)}(x - x_k) - b_y^{(k)}(y - y_k) - H_0, \quad (8)$$

Where $(x_k, y_k)$ denotes the center of the cell $P_k$, and $h^{(k)} = h(x_k, y_k)$.

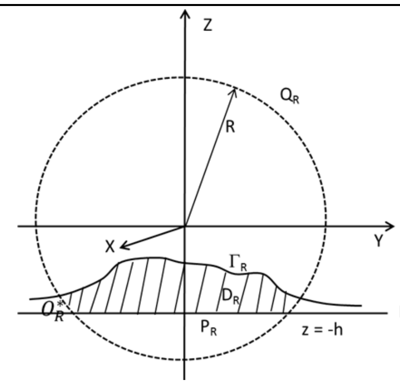


Figure 1: An illustration of the contrast density model for a sediment-basement interface. The plane P is the average depth of the sediment-basement interface, and $\Gamma_R$ is the actual sediment-basement interface.

In a discretized form, equations (6) and (7) can be represented as follows:

$$g_\alpha(\mathbf{r}'_n) = \sum_{k=1}^{N_m} f_{\alpha\gamma}^{(nk)} h^{(k)} b_\gamma^{(k)}, \quad (9)$$

where:

$$f_{\alpha\gamma}^{(nk)} = -\gamma_g\rho_0\Delta_{\alpha z\gamma\eta} \frac{\left(r_\eta^{(k)} - r_\eta^{(k)'}\right)}{\left|\mathbf{r}^{(k)} - \mathbf{r}'_n\right|^3} b_\gamma \Delta_x \Delta_y, \quad (10)$$

and:

$$r_x^{(k)} = x_k, r_y^{(k)} = y_k, r_z^{(k)} = z_k = h^{(k)} - H_0, \quad (11)$$
$$r_x^{(n)'} = x^{(n)'}, r_y^{(n)'} = y^{(n)'}, r_z^{(n)'} = z^{(n)'}.$$

As usual, the twice recurring index in equation (10) indicates the summation over this index. Similarly, the discretized form of the equation for the gravity gradient fields can be expressed as follows:



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$$g_{\alpha v}(\mathbf{r}_n') = \sum_{k=1}^{N_m} F_{\alpha v \gamma}^{(nk)} h^{(k)} b_{\gamma}^{(k)}, \qquad (12)$$

where:

$$F_{\alpha v \gamma}^{(nk)} = \frac{\gamma_g \rho_0 \Delta_{\alpha z \gamma \eta}}{|\mathbf{r}^{(k)} - \mathbf{r}_n'|^5} \left[ 3 \left( r_v^{(k)} - r_v^{(n)\prime} \right) \left( r_v^{(k)} - r_v^{(n)\prime} \right) - \left| \mathbf{r}^{(k)} - \mathbf{r}_n' \right|^2 \delta_{v\eta} \Delta_x \Delta_y. \right.$$

**Inversion for a density contrast surface based on 3D Cauchy-type integrals**

In our forward modeling process, the model parameters are the elevations, $h^{(k)} = h(x_k, y_k)$, of the density contrast surface with respect to the horizontal plane P. As we can see from the forward modeling equations, the forward operator is nonlinear. Correspondingly, the inversion is also a nonlinear problem. The traditional inversion of potential field data for the density distribution is a linear problem, and the Fréchet derivative can be easily found and it does not change during the iterative inversion. In our inversion, the Fréchet derivative is a function of model parameters which can be expressed in analytical form.

The inversion is based on the minimization of the Tikhonov parametric functional:

$$\mathrm{P}^{\alpha}(\mathbf{m}, d) = (\mathbf{W}_{\mathrm{d}}\mathbf{A}\mathbf{m} - \mathbf{W}_{\mathrm{d}}\mathbf{d})^T (\mathbf{W}_{\mathrm{d}}\mathbf{A}\mathbf{m} - \mathbf{W}_{\mathrm{d}}\mathbf{d}) \qquad (13)$$
$$+\alpha\left(\mathbf{W}_{\mathrm{e}}\mathbf{W}_{\mathrm{m}}\mathbf{m} - \mathbf{W}_{\mathrm{e}}\mathbf{W}_{\mathrm{m}}\mathbf{m}_{apr}\right)^T \left(\mathbf{W}_{\mathrm{e}}\mathbf{W}_{\mathrm{m}} - \mathbf{W}_{\mathrm{e}}\mathbf{W}_{\mathrm{m}}\mathbf{m}_{apr}\right)$$
$$\rightarrow min$$

where $\mathrm{W_d}$ is the data weighting matrix; m is the vector of the model parameters, **h**; and $\mathbf{W}_{\mathrm{m}}$ is a diagonal matrix of the model parameters weights based on integrated sensitivity:

$$\mathbf{W}_{\mathrm{m}} = \boldsymbol{diag}(\mathbf{F}^{\mathrm{T}}\mathbf{F})^{1/4} \qquad (14)$$

Matrix $\mathrm{W_e}$ is also a diagonal matrix of the minimum support stabilizer providing focusing inversion:

$$\mathbf{W}_{\mathrm{e}} = \boldsymbol{diag}[w_e] = \boldsymbol{diag}\left[\frac{1}{(m_i^2 + e^2)}\right]. \qquad (15)$$

The minimization of the Tikhonov parametric functional is based on the reweighted regularized conjugate gradient method (Zhdanov, 2002).

The developed theory and method have been implemented in the computer code which was tested on synthetic gravity and gravity gradient data for Ensenada Bay.

**Numerical example for Ensenada Bay gravity and gravity gradient data**

In this section, we present a model study for the inversion of the gravity and gravity gradient data for the sediment-basement interface. In our model, we assume that the horizontal reference plane *P* is located at a depth of 1000 m under the earth's surface. Our model is a reconstructed model of Ensenada Bay basement from Gallardo-Delgado et al., (2002). For this model, the left portion of the sediment-basement interface is above the reference plane *P*, while the right portion is below plane *P*. The actual interface extends from -1400 m to -600 m in the z direction. For this model, the vertical gravity component and full tensor gravity gradiometry data were computer simulated.

Figure 2 shows the basement, the sedimentary basin, and the bathymetry of Ensenada Bay. Figure 3 is a 3D view of the model that we built based on the 2D vertical geological section. Figure 4 shows the observed data ($g_z$ and full tensor gravity gradiometry) with 5% random noise added. Figure 5 presents a 3D view of the inversion result for this model, and Figure 6 shows the 2D vertical section of the interface at $y = 0$. One can see that the reconstructed model for Ensenada Bay is well recovered from the inversion of both the vertical gravity field component and of the full tensor gravity gradient data.

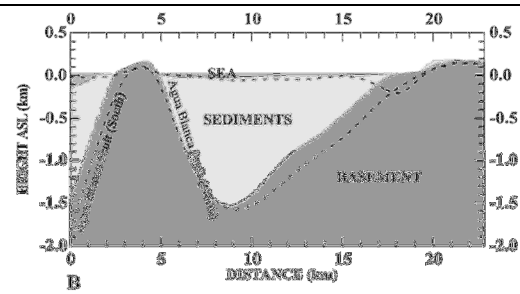


Figure 2: 2D cross section showing the basement, sedimentary basin, and bathymetrytopography of Ensenada Bay (after Gallardo-Delgado, 2002).

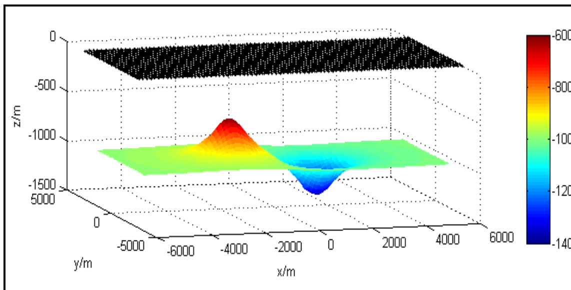


Figure 3: Model 3. A realistic model of sediment-basement interface with complex anomaly. The surface in the figure is the actual boundary for sediment and basement with maximum/minimum value of —600/-1400 m and the reference sediment-basement interface is a horizontal plane located at -1000 m. The black dots are the observation stations for recording the gravity response.




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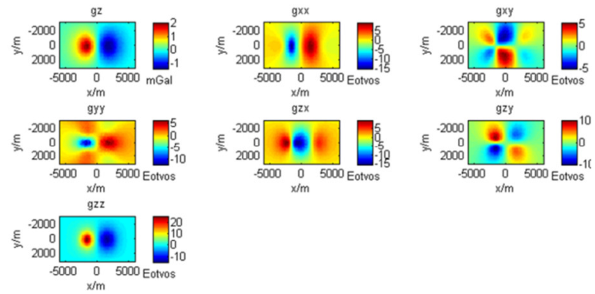


Figure 4: Model 3: Synthetic gravity and full tensor gravity gradiometry data with 5% random noise added.

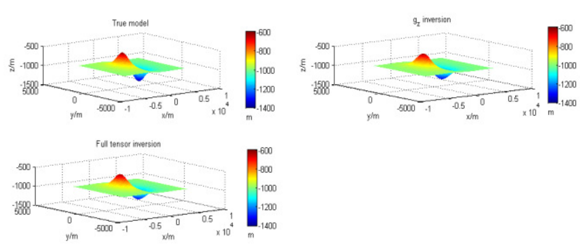


Figure 5: The upper left panel shows the true model; the upper right panel shows the inversion result from $g_z$; the bottom left panel shows the full tensor inversion result.

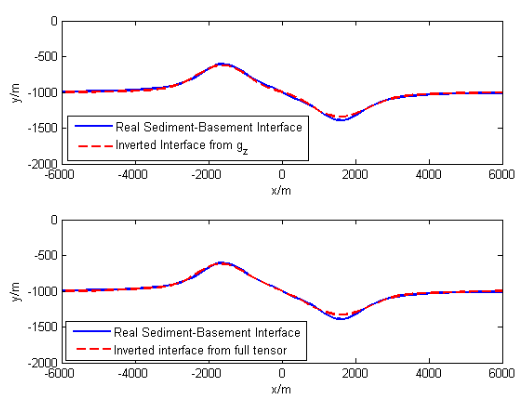


Figure 6: The true profile of the sediment-basement interface at y=0 (blue line) and the inversion results. The upper panel shows a comparison between the true model and the inversion result from $g_z$. The bottom panel shows a comparison between the true model and the inversion result from full tensor gravity gradient.

## Conclusions

In this paper, we have introduced the application of 3D Cauchy-type integrals for numerical modeling and inversion of gravity and gravity gradiometry data. We considered the case of the sediment-basement interface with a contrast density. In our modeling process, we need to discretize the density contrast surface only, which resulted in significant reduction of the amount of computations compared to the traditional methods based on volume discretization. We have also considered an inverse problem to recover the location of a density contrast surface.

The inverse problem is a non-linear and ill-posed problem. The Fréchet derivative matrix is computed analytically by the differentiation of the forward modeling operator. We consider a minimization of Tikhonov parametric functional in order to solve this ill-posed inverse problem. Our model study demonstrates that both the shape and location of the density contrast surface can be recovered well by our method.

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**EDITED REFERENCES**

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